Exam II: Determinants, Eigenvalues and Vector Spaces

1. Determinants

(a) Computational Issues:
   i. Be able to compute the 2×2 det.
   ii. Be able to compute an $n \times n$ det. using cofactor expansions.
   iii. For upper (lower) triangular matrices, the determinant is the product of the diagonal elements.
   iv. Know the determinants of the elementary matrices corresponding to the elementary row operations.
   v. Be able to compute an $n \times n$ determinant by first row reducing to upper triangular.
   vi. Computational Theorems: (i.e., problems 39, 40 - p. 194)
      - If A is $n \times n$, then $\det(k \cdot A) = k^n \det(A)$.
      - If A, B are $n \times n$, then $\det(AB) = \det(A) \cdot \det(B)$.
      - If A is $n \times n$, then $\det(A^T) = \det(A)$.

(b) Conceptual Issues:
   i. Know the effects of the three elementary row operations on the determinant.
   ii. Know the relationship between invertibility and the determinant.

2. Eigenvalues and Eigenvectors

(a) Definition: $A\boldsymbol{v} = \lambda\boldsymbol{v}$

(b) Computation (or to show properties):

$$(A - \lambda I)\boldsymbol{v} = 0 \Leftrightarrow \det(A - \lambda I) = 0$$

First compute the eigenvalues, then for each of those, solve for the corresponding eigenvectors. We will only work with real eigenvalues for this exam.

(c) Properties (Each of these can be shown using the properties of determinants. For practice, try it yourself!)
   - $\lambda = 0$ if and only if A is not invertible.
   - If λ is an eigenvalue of A, it is an eigenvalue of A^T.
   - If $A = PBP^{-1}$, then A and B have the same eigenvalues.
   - If A has n distinct eigenvalues, the eigenvectors are linearly independent (and thus form a basis for $\mathbb{R}^n$).

3. Vector Spaces

(a) Definitions associated with General Vector Spaces.
   - Vector Space (Be able to check if something is a vector space, given the 10 axioms. You do not have to memorize the 10 axioms.)
   - Subspace
   - Spanning Set
   - Kernel (Also contrast a Kernel with a Null Space)
   - Linear Transformation, $1-1$, Onto
   - Linear Independence
   - Basis
   - $\mathcal{B}$ -Coordinates for x and the definitions of $[x]_{\mathcal{B}}, P_{\mathcal{B}}$.

- Isomorphism
- Dimension of a vector space (Including the special cases of zero and infinity)
- The rank of a matrix.

(b) The Four Fundamental Subspaces: Col(A), Null(A), Row(A), Col(A^T).

(c) Specific Examples:

- $\mathbb{R}^n$, $\mathbb{P}^n$, $M_{m\times n}$, $C[a,b]$
- Standard Basis for $\mathbb{R}^n$, Standard Basis for $\mathbb{P}^n$

(d) Theorems for Vector Spaces (Note: These are summarized below, you should check each one to be sure you understand the specific conditions):

- Theorem 1: Span $\{v_1, \dots v_p\}$ is a subspace.
- The Spanning Set Theorem: If we remove a linearly dependent vector from a spanning set, the span is not changed. This process can be continued to find a Basis.
- Theorem 7: The coordinate representation for a vector exists and is unique. (It exists because a basis *spans* the subspace, it is unique because a basis has *linearly independent* vectors.
- Theorem 8: If a vector space V has a basis consisting of n vectors, then V is *isomorphic* to $\mathbb{R}^n$. Know what the isomorphism is.
- Theorem 10: If one basis of V contains n vectors, then EVERY basis of V contains exactly n vectors. (This is clear if we think of V as being isomorphic to $\mathbb{R}^n$, and creates the concept of Dimension).
- Theorem 12 (Finding a basis): If V is $p-$dimensional, then *any* set of p linearly independent vectors will form a basis for V. (Note: This also covers Theorem 11).

(e) Theorems specific to a matrix A:

- Theorems 2,3: Null(A) and Col(A) are subspaces.
- Theorem 6 (Finding a basis for Col(A)): The pivot columns of A form a basis for Col(A).
- Theorem 13 (Finding a basis for Row(A)): If B is the EF of A, the nonzero rows of B forms a basis for Row(A). Contrast this with finding a basis for Col(A).
- Theorem 14: $\dim(\text{Col}(A)) = \dim(\text{Row}(A))$. This also implies that the $\dim(\text{Col}(A)) = \dim(\text{Col}(A^T))$. This gives the dimensions in the table explaining the dimensions of the four fundamental subspaces.
- The Rank Theorem: $\text{rank}(A) + \dim(\text{Null}(A)) = n$

(f) Computational Issues

- Solve problems like: Is w in the Span($v_1, \dots, v_p$)?
- Compute a basis for Null(A), Col(A), Row(A).
- Go from a spanning set to a basis.
- Understand and use:
$$[x]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\boldsymbol{x} \text{ and } \boldsymbol{x} = P_{\mathcal{B}}[x]_{\mathcal{B}}$$
- Determine if a given vector is in one of the four fundamental subspaces.
- Compute the dimension (and basis) for a subspace.
- Know how to compute the dimensions of the four fundmental subspaces.