Solutions to Review Questions

(1) The area of the parallelogram is the determinant of the matrix formed by: $[u, v]$: 4.

(2) If det A is 5, det B is 5 (Row operations of that type do not effect the determinant). det C is $2 \cdot 5 = 10$, so det BC is 50.

(3) Basis for the Column Space: The first, second and fourth vectors of A. (Be sure you write the columns of A, not B).

Basis for Row A: The first, second and third rows of B

Basis for Null A: $(-4, 3, 1, 0, 0)^T, (-3, -5, 0, 4, 1)^T$

(4) (a) dim of Col A is k, by definition. (b) dim of Row A is also k (a main theorem), (c) dim Null A is $n-k$, since Row A and Null A are in R^n, (d) dim Null A^T is $m-k$.

(5)
- The first H is not a subspace of R^3 because it is not closed under scalar multiplication (for example, $c = -1$).
- The second H is a subspace because it consists of the spanning set of $(1, 1, 2, 4)^T, (3, -1, 1, 0)^T$.
- The third H is a subspace, which can be verified directly:
  - The zero function satisfies the equation.
  - Let $f, g \in H$. Then $f' = f$ and $g' = g$, so $(f+g)' = f' + g' = f + g$.
  - Let $f \in H$. Then $f'' = f$, and $(cf)'' = cf'' = cf$.
- The fourth H is also a subspace, since it is spanned by $(1, 1, 0)^T$ and $(1, 0, 1)^T$
- We know that H_λ is a subspace (it was also a group quiz):
  - $0 \in H_\lambda$, since $A0 = \lambda \cdot 0$.
  - Let $u, v \in H_\lambda$. Then $Au = \lambda u$ and $Av = \lambda v$. Therefore, $A(u + v) = Au + Av = \lambda u + \lambda v = \lambda(u + v)$.
  - Let $u \in H_\lambda$. Then $Au = \lambda u$, and $A(cu) = cAu = c\lambda u = \lambda(cu)$.

  Note that this shows that any scalar multiple of an eigenvector is an eigenvector, and any linear combination of eigenvectors (for one eigenvalue) is also an eigenvector.

(6) We first re-write what we're given: Let A be a 7×9 matrix whose nullspace has dimension 2. Will the mapping $x \to Ax$ be onto? The answer is yes if and only if the dimension of the columnspace of A is 7. Since the dimension of the nullspace of A is 2, the dimension of the rowspace must be 7. The dimensions of the rowspace and columnspace are the same, so the dimension of the columnspace of A is also 7. Therefore, the range is all of R^7, and the mapping is onto.

(7) We know that

$$\dim(\text{Col}(A)) + \dim(\text{Null}(A)) = \#\text{ cols of } A$$

which is the Rank Theorem. Replace A by A^T to get:

$$\dim(\text{Col}(A^T)) + \dim(\text{Col}(A)) = m$$

and we use the following to replace $\dim(\text{Col}(A^T))$:

$$\dim(\text{Col}(A^T)) = \dim(\text{Row}(A^T)) = \dim(\text{Col}(A))$$

(8) First, $Ax = b$ has a solution for every $b \in R^m$ if and only if the columnspace of A is all of $\mathbb{R}^m$. This means that $\dim(\text{Col}(A)) = m$. This will be true (by

the previous exercise) if and only if $\dim(\text{Null}(A^T)) = 0$. This will be true if and only if the only solution to $A^T y = 0$ is the trivial solution, $y = 0$.

(9) Find the eigenvalues and eigenvectors for the matrix A below:

(NOTE: The matrix on the question sheet didn't work out nicely. Replace that matrix with the following, and re-do.)

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

In this case, you find that the eigenvalues are $1, -2, -2$ and the corresponding eigenvectors are: $(1, -1, 1)^T, (-1, 1, 0)^T$ and $(-1, 0, 1)^T$.

(10) $\boldsymbol{u}, \boldsymbol{v}$ were unfortunately left off this question. Try it with the following $\boldsymbol{u}, \boldsymbol{v}$:

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

Now to answer the question:

If $\boldsymbol{x} = (b_1, b_2, b_3)^T$ is a point on the plane spanned by $\boldsymbol{u}, \boldsymbol{v}$, then there exists a solution to

$$\boldsymbol{x} = c_1\boldsymbol{u} + c_2\boldsymbol{v} \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & b_1 \\ 1 & -1 & b_2 \\ 1 & 0 & b_3 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & b_1 \\ 0 & -4 & -b_1 + b_2 \\ 0 & -3 & -b_1 + b_3 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & b_1 + \frac{3}{4}(-b_1 + b_2) \\ 0 & 1 & -\frac{1}{4}(-b_1 + b_2) \\ 0 & 0 & -\frac{3}{4}(-b_1 + b_2) - b_1 + b_3 \end{array}\right]$$

Since this must have a solution, the third part must be zero:

$$-\frac{3}{4}(-b_1 + b_2) - b_1 + b_3 = 0 \Rightarrow \frac{1}{4}b_1 - \frac{3}{4}b_2 + b_3 = 0 \Rightarrow b_1 - 3b_2 + 4b_3 = 0$$

(This was not part of the question, but you might try to verify that this is indeed the equation of the plane).

The coordinates of $\boldsymbol{x}$ with respect to $\boldsymbol{u}, \boldsymbol{v}$ are the first two values,

$$[\boldsymbol{x}]_{\{u,v\}} = ((1/4)b_1 + (3/4)b_2, (1/4)b_1 - (1/4)b_2))^T$$

and this gives the isomorphism:

$$(b_1, b_2, b_3)^T \longrightarrow (1/4)(b_1 + 3b_2, b_1 - b_2)^T$$

(11) Show that the range of T is a subspace, where T is a linear mapping:

- $0 \in T(V)$, since $T(0) = 0$, and $0 \in V$.
- Let y_1, y_2 be in $T(V)$. Then there exists vectors $u, v \in V$ such that $y_1 = T(u)$ and $y_2 = T(v)$. Now, we have that $y_1 + y_2 = T(u) + T(v) = T(u + v)$, so $y_1 + y_2$ is in $T(V)$.
- Let $y \in T(V)$. Then there exists $u \in V$ such that $y = T(u)$. Now, $cy = cT(u) = T(cu)$, so $cy \in T(V)$.

(12) Let A be an $n \times n$ matrix. A is invertible if and only if (a) $\text{Null}(A) = \{0\}$, (b) The columns of A form a basis for R^n, (c) The rank of A is n.

(13) We have:

$$x = P_B[x]_B, \quad \text{and} \quad [x]_B = P_B^{-1}x$$

where

$$P_B = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad P_B^{-1} = \frac{-1}{7}\begin{bmatrix} -1 & -3 \\ -2 & 1 \end{bmatrix}$$

So if $x = (3,0)^T$, $[x]_B = (\frac{3}{7}, \frac{6}{7})^T$ (Note that you can verify this by multiplying $P_B[x]_B$ to see if you get x back again).

If $[x]_B = (4,-2)^T$, then $x = (-2,10)^T$.

(14) Recall that in Z_2 arithmetic, $1+1=0$. Perform the following row ops to get the matrix that follows: $r_1 + r_3 \to r_3$, $r_2 + r_3 \to r_3$, $r_3 + r_2 \to r_2$. This leaves the first three columns as pivot columns, the last two columns represent two free variables:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

A basis for the columnspace is formed by taking the first three columns of the original matrix A. A basis for the nullspace is: $(1,0,1,1,0)^T, (1,1,0,0,1)^T$. The rank is 3, which in this case is the same as if we did normal arithmetic. If we use normal arithmetic, the matrix reduces to:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

Again, the basis for the columnspace is formed by taking the first three columns of A. A basis for the nullspace is now: $(-1,0,-1,1,0)^T, (-1,1,-2,0,1)^T$

(15) There was a typo in the questions: The vectors $\boldsymbol{v}$ should be indexed as $1,2,3$. Also, the last question is to compute the coordinates of $\boldsymbol{w}$ with respect to $\mathcal{B}$

The answer to the first question is NO, since $w \neq v_1, w \neq v_2$, and $w \neq v_3$. (This question is about notation!) To answer the second question, we row reduce the augmented matrix to get:

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + (1-2t)\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + t\begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

or the coordinates of $\boldsymbol{w}$ are $(1, 1-2t, t)$ for any t. Note that, because $\mathcal{B}$ is not a basis, then there are an infinite number of possible coordinates.

(16) (a) $(2)(-3) = -6$, (b) $\frac{1}{2}$, (c) $5^4(-3)$, (d) Cannot be computed. (e) -3

(17) In general, an isomorphism is a $1-1$, onto, linear mapping. In the case of the theorem, the isomorphism is the coordinate mapping:

$$x_1 v_1 + \ldots + x_n v_n \in V \quad \leftrightarrow \quad (x_1, \ldots, x_n)^T \in R^n$$

(18) The system is describing $A\boldsymbol{x} = \boldsymbol{0}$, where A is 10×12. We could translate the question to be about the dimension of the nullspace of A- Is it possible for the nullspace of A to be 1?

Since there are 12 variables but only 10 equations, we must have at least two free variables, and so the nullspace must be spanned by at least two vectors. Therefore, it is not possible for all solutions to $A\boldsymbol{x} = \boldsymbol{0}$ to be written as multiples of *one* nonzero vector.

(19) See the picture.

(20) We know that:

$$\boldsymbol{x} = P_B[x]_B, \text{ and } \boldsymbol{x} = P_C[x]_C$$

where:

$$P_B = \begin{bmatrix} 3 & 4 \\ -5 & -6 \end{bmatrix}, P_c = \begin{bmatrix} 4 & 6 \\ 5 & 7 \end{bmatrix}$$

Now, if we are given $[x]_B$, we could multiply it my P_B to get $\boldsymbol{x}$. Using the second equation, multiply that by P_c^{-1} to get $[x]_c$. Therefore,

$$[x]_C = P_C^{-1} P_B [x]_B$$

Similarly,

$$[x]_B = P_B^{-1} P_c [x]_C$$

where:

$$P_B^{-1} P_C = \begin{bmatrix} -22 & -32 \\ \frac{35}{2} & \frac{51}{2} \end{bmatrix}, \qquad P_C^{-1} P_B = \begin{bmatrix} \frac{-51}{2} & -32 \\ \frac{35}{2} & 22 \end{bmatrix}$$

(21) Let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be a basis for H. We know that $T(H)$ is a subspace by problem 11 (or show it directly). Now, consider the dimension of $T(H)$. Since $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is a basis for H, $\{T(\boldsymbol{v}_1), \ldots, T(\boldsymbol{v}_n)\}$ is a spanning set for $T(H)$. Since we cannot say if $\{T(\boldsymbol{v}_1), \ldots, T(\boldsymbol{v}_n)\}$ is a linearly independent set, the dimension of $T(H)$ is less than or equal to n. The dimension will be n only if the set $\{T(\boldsymbol{v}_1), \ldots, T(\boldsymbol{v}_n)\}$ is linearly independent, which we can guarantee only if T is a $1-1$ mapping.

(22) Both questions can be answered at once doing the row reduction on the matrix below.

Given $3 - t + 2t^2$ in standard coordinates is like being given the vector $(3, -1, 2)^T$ in standard coordinates. Being asked what the new coefficients are is like being asked to translate $(3, -1, 2)^T$ into the new coordinates. Therefore, the answer is the solution to the matrix equation:

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 4 & 2 \end{array}\right]$$

which is $(4, \frac{-1}{2}, \frac{1}{2})^T$, or:

$$4(1) + \frac{-1}{2}(2t) + \frac{1}{2}(-2 + 4t^2) = 3 - t + 2t^2$$

(23) Problem 1: We should be writing these as COLUMNS if we are going to correctly interpret the row reduction. However, we can still do an interpretation of the rows.

What is being done here is equivalent to finding a basis for the Row space of the matrix. Here, we see that the Row space has two basis vectors- Therefore, the three original "rows" can be written as a linear combination of these two vectors, so that the original set of polynomials could not be linearly independent (that is, three combinations of two vectors cannot give you three independent vectors).

(24) The rank is 0 if $a = b = c = 0$, otherwise the rank is 1 (we can see that by multiplying out, then row reducing $\boldsymbol{u}\boldsymbol{v}^T$.