MATH 300, Second Exam REVIEW QUESTIONS

You should also look at the supplementary exercises for Chapter 4, problems 1-11

1. Let $\boldsymbol{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, Let S be the parallelegram with vertices at $\mathbf{0}, \boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{u} + v$. Compute the area of S.

2. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $B = \begin{bmatrix} a+2g & b+2h & c+2i \\ d+3g & e+3h & f+3i \\ g & h & i \end{bmatrix}$, and $C = \begin{bmatrix} g & h & i \\ 2d & 2e & 2f \\ a & b & c \end{bmatrix}$.

   If $\det(A) = 5$, find $\det(B), \det(C), \det(BC)$.

3. Assume that A and B are row equivalent, where:

$$A = \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ -2 & -3 & 1 & -1 & -5 \\ -3 & -4 & 0 & -2 & -3 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 4 & 0 & 3 \\ 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) Find a basis for Col(A)

   (b) Find a basis for Row(A)

   (c) Find a basis for Null(A)

4. Let A be an $m \times n$ matrix of rank k. What are the dimensions of the 4 fundamental subspaces?

5. Determine if the following sets are subspaces of V. Justify your answers.

$$H = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, a \geq 0,\ \ b \geq 0,\ \ c \geq 0 \right\}, \qquad V = \mathbb{R}^3$$

$$H = \left\{ \begin{bmatrix} a+3b \\ a-b \\ 2a+b \\ 4a \end{bmatrix},\ \ a, b \text{ in } \mathbb{R} \right\}, \quad V = \mathbb{R}^4$$

$$H = \{f : f'(x) = f(x)\}, V = C^1[\mathbb{R}]$$

   (C^1 is the space of differentiable functions where the derivative is continuous).

   H is the set of vectors in $\mathbb{R}^3$ whose first entry is the sum of the second and third entries, $V = \mathbb{R}^3$.

   For the following question, assume A is $n \times n$, and let λ be a given eigenvalue of A. Show directly (without the use of theorems) that the following is a subspace of $\mathbb{R}^n$:

$$H_\lambda = \{\boldsymbol{x} \ : \ A\boldsymbol{x} = \lambda \boldsymbol{x}\}$$

6. Consider a homogeneous system of seven linear equations in nine variables. Suppose that there are two solutions of the system that are not multiples of one another, and all other solutions are linear combinations of these solutions. Will the associated nonhomogeneous system have a solution for every possible choice of constants on the right sides of the equations? Justify your answer carefully.

7. Let A be $m \times n$. Justify: $\dim(\text{Col}(A)) + \dim(\text{Null}(A^T)) = m$

8. Use the previous answer to justify the following statement:

   $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for every $\boldsymbol{b} \in \mathbb{R}^m$ if and only if the equation $A^T\boldsymbol{y} = \mathbf{0}$ has only the trivial solution.

9. Find the eigenvalues and eigenvectors for the matrix A below:

$$A = \begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

10. Let $\boldsymbol{u}, \boldsymbol{v}$ be as defined below. The span of these vectors forms a plane in $\mathbb{R}^3$. (a) If $\boldsymbol{x} = (b_1, b_2, b_3)^T$, is a point on the plane, find $[x]_{\{u,v\}}$ in terms of b_1, b_2, b_3. (b) We said that a plane through the origin is isomorphic to $\mathbb{R}^2$. What is the isomorphism? (Assume that the domain is the set of points (b_1, b_2, b_3) on the plane).

11. Prove that, if $T : V \mapsto W$ is a linear transformation between vector spaces V and W, then the range of T is a subspace of W.

12. Let A be an $n \times n$ matrix. Write statements from the Invertible Matrix Theorem that are each equivalent to the statement "A is invertible". Use the following concepts, one in each statement: (a) Null(A) (b) Basis (c) Rank

13. Let $\mathcal{B} = \{u_1, u_2\}$, where
$$u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, u_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$
If $\boldsymbol{x} = (3, 0)$, what is $[x]_\mathcal{B}$? If $[x]_\mathcal{B} = (4, -2)$, what is $\boldsymbol{x}$?

14. Find a basis for the null space and column space of A using Z_2 arithmetic. Determine the rank of the matrix (again, with respect to Z_2). Answer the same questions, but use the standard real number arithmetic.
$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

15. Let $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \boldsymbol{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$. Is $\boldsymbol{w}$ in $\mathcal{B} = \{\boldsymbol{v}_\infty, \boldsymbol{v}_\in, \boldsymbol{v}_\ni\}$? Is $\boldsymbol{w}$ in the span of these vectors? If so, compute $[x]_\mathcal{B}$ $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$?

16. If A, B are 4×4 matrices with $\det(A) = 2$ and $\det(B) = -3$, what is the determinant of the following (if you can compute it): (a) $\det(AB)$, (b) $\det(A^{-1})$, (c) $\det(5B)$, (d) $\det(3A - 2B)$, (e) $\det(B^T)$

17. We said that, if V is a vector space with n basis vectors, then V is isomorphic to $\mathbb{R}^n$. What is the isomorphism? In general, what is an isomorphism?

18. Is it possible that all solutions of a homogeneous system of ten linear equations in twelve variables are multiples of one fixed nonzero solution? Discuss.

19. Let $\mathbb{R}^n$ be represented by three sets of basis vectors, $\mathcal{B} = \{\boldsymbol{b}_i\}_{i=1}^n$, $\mathcal{C} = \{\boldsymbol{c}_i\}_{i=1}^n$, and the standard basis $\{\boldsymbol{e}_i\}_{i=1}^n$. Draw a diagram that relates these three versions of $\mathbb{R}^n$, and write down the mappings that relate each space to the other two.

20. Let $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix} \begin{bmatrix} 4 \\ -6 \end{bmatrix} \right\}$, and $\mathcal{C} = \left\{ \begin{bmatrix} 4 \\ 5 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix} \right\}$. Write down the matrices that take $[x]_C$ to $[x]_B$ and from $[x]_B$ to $[x]_C$.

21. Let H be a subspace of V, and let T be a linear mapping of vector space V to vector space W, where V, W are finite dimensional. Show that $T(H)$ is a subspace of W, and show that $\dim(T(H)) \leq \dim(H)$. What do we need to say about T in order to get equality?

22. (a) Show that $\{1, 2t, -2 + 4t^2\}$ is a basis for P_3. (b) Find the coordinates of $3 - t + 2t^2$ in terms of this basis.

23. What is wrong with the following:

Question: Does the set $p_1(t) = 1 - t$, $p_2(t) = t^2 - 3$, $p_3(t) = t^2 - 3t$. Show that p_1, p_2, p_3 are linearly independent.

Solution:

p_1 can be written as $[1, -1, 0]$, p_2 as $[-3, 0, 1]$, and p_3 as: $[0, -3, 1]$. Therefore, we row reduce the matrix:

$$\begin{bmatrix} 1 & -1 & 0 \\ -3 & 0 & 1 \\ 0 & -3 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 0 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

There is a row of zeros, and a free variable. Therefore, the functions p_1, p_2, p_3 are linearly dependent.

24. Show that the rank of $\boldsymbol{u}\boldsymbol{v}^T \leq 1$ if $\boldsymbol{u} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. (Hint: Multiply out the matrix).