Concept Relationships-Solutions to the Questions

1. What is the definition of the span of a set of vectors?

   The span of a set of vectors is the set of all linear combinations of those vectors.

2. What is the meaning behind "The set of vectors spans $\mathbb{R}^m$"?

   To say that the set of vectors spans $\mathbb{R}^m$ means that any vector in $\mathbb{R}^m$ can be written as a linear combination of those vectors.

   When we talked about the columns of a matrix spanning $\mathbb{R}^m$, we meant that there was always a solution to $A\boldsymbol{x} = \boldsymbol{b}$, for all $\boldsymbol{b} \in \mathbb{R}^m$.

   NOTE ON LANGUAGE: This definition implies that the vectors are indeed elements of $\mathbb{R}^m$ to begin with. For example, vectors in $\mathbb{R}^3$ cannot span $\mathbb{R}^2$, because they are not elements of $\mathbb{R}^2$. In general, vectors in $\mathbb{R}^m$ cannot span $\mathbb{R}^n$ if $m \neq n$.

3. Can a set of n vectors span $\mathbb{R}^m$ if $m > n$? If $m < n$?

   If the matrix formed from the vectors is "wide", it is possible for these column vectors to span $\mathbb{R}^m$. This would be if $m < n$.

   If the matrix formed from the vectors is "tall", it is impossible for the column vectors to span $\mathbb{R}^m$, since we cannot have a pivot in each row. This would be for $m > n$

4. What is the definition of linear independence in a set of vectors?

   A set of vectors, $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots \boldsymbol{v}_k\}$ is linearly independent if the only solution to the equation:

   $$c_1\boldsymbol{v}_1 + \dots c_k\boldsymbol{v}_k = 0$$

   is $c_1 = c_2 = \dots = c_n = 0$, which is the trivial solution.

   This definition can be tied to solutions of $A\boldsymbol{x} = \boldsymbol{0}$ by taking $A = [\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k]$.

5. What is the heuristic definition of linear independence?

   A set of vectors is linearly independent if no vector is a linear combination of the other vectors.

6. Can a set of n vectors in $\mathbb{R}^m$ be linearly independent if $m > n$? If $m < n$?

   If the matrix formed by the n column vectors is "wide" $(m < n)$, then we must have free variables in the RREF of the matrix. This means that $A\boldsymbol{x} = \boldsymbol{0}$ has nontrivial solutions, so its columns must be linearly dependent.

   If the matrix formed by the n column vectors is "tall" $(m > n)$, then it is possible to have a pivot in each column, which means that it is possible for the columns of the matrix to be linearly independent.

7. Show that (IV) implies (I), (II), (III).

   Since we've gone through the arguments several times, here's the "shorthand" version:

   Pivot in each row $\Rightarrow$ We cannot have a row of zeros in the RREF of A $\Rightarrow$ There is a solution to $A\mathbf{x} = \boldsymbol{b}$ for every $\boldsymbol{b}$ $\Rightarrow$ The cols of A span $\mathbb{R}^m$ $\Rightarrow$ By the definition of $A\boldsymbol{x}$, every $\boldsymbol{b}$ can be written as a linear combination of the cols of A.

8. Show that (VIII) implies (V), (VI), (VII).

   Here is a "shorthand" version- You would want to write them out:

   Pivot in each column $\Rightarrow$ No free variables $\Rightarrow$ If $A\boldsymbol{x} = \boldsymbol{b}$ has a solution, it is unique $\Rightarrow$ $A\boldsymbol{x} = \boldsymbol{0}$ has only the trivial solution $\Rightarrow$ The only solution to $c_1\boldsymbol{a}_1 + \dots + c_n\boldsymbol{a}_n = 0$ is the trivial solution $\Rightarrow$ The cols of A are linearly independent.

9. In the case that A is square, why does any of (I)-(IV) imply that all 8 items are true? Does the same argument hold true if we only know that one of (V)-(VIII) hold?

   Mainly because, if A is square, then "pivot in every row" implies "pivot in every column".

   For the following two questions, T is the linear mapping, and the matrix A is the realization of T (so that $T(x) = A\mathbf{x}$)

10. If A is $m \times n$, $m > n$, Can T be $1-1$? Can T be onto?

    If $m > n$, then A is a "tall" matrix. By our answers to problems 3 and 6, we cannot have a pivot in every row (so T cannot be onto), but we can have a pivot in every column (so T might be $1-1$).

11. If A is $m \times n$, $m < n$, Can T be $1-1$? Can T be onto?

    If $m < n$, then A is a "wide" matrix. In this case, it is impossible to have a pivot in every column, so T cannot be $1-1$. However, it is possible to have a pivot in every row, so T might be "onto".

12. Consider the statement: If AB is invertible, then B is invertible. (i) Show that this is FALSE in general (Come up with two matrices A, B). (ii) If A, B are $n \times n$, show that the statement is TRUE.

    You can come up with a lot of matrices A, B. A matrix we've looked at previously, with A wide, required that there was a pivot in every row (and also we want free variables). From that, you can row reduce $[A \mid I]$ to solve the matrix equation $AB = I$ for B.

    If A, B are $n \times n$, we can use the Invertible Matrix Theorem. By the IMT, if AB is invertible, there exists a matrix W so that $W(AB) = I$. From the properties of matrix multiplication, we can write this as $(WA)B = I$, which says there is a matrix $D = WA$ so that $DB = I$, and so B is invertible by the IMT.

13. Show that, if T is a linear mapping and the only solution to $T(x) = 0$ is the trivial solution, then T must be $1-1$ (NOTE: Use arguments applicable to *functions*, not just matrices).

    We are told that the only solution to $T(\boldsymbol{x}) = \mathbf{0}$ is the trivial solution. To show that T is $1-1$, we must show that, if $\boldsymbol{x}_1 \neq \boldsymbol{x}_2$, then $T(\boldsymbol{x}_1) \neq T(\boldsymbol{x}_2)$.

    Given that: $\boldsymbol{x}_1 \neq \boldsymbol{x}_2$, we know that $\boldsymbol{x}_1 - \boldsymbol{x}_2 \neq \mathbf{0}$. Since $T(\boldsymbol{x}) = 0$ has only the trivial solution, this implies that $T(\boldsymbol{x}_1 - \boldsymbol{x}_2) \neq 0$. By the linearity of T, this implies that $T(\boldsymbol{x}_1) - T(\boldsymbol{x}_2) \neq 0$, and so $T(\boldsymbol{x}_1) \neq T(\boldsymbol{x}_2)$

14. Show (by using the definition of linear maps) that, if T is invertible, then the inverse, S is also linear.

    Let S be the inverse of T. We want to show that S is linear- that is,

    $$S(u + v) = S(u) + S(v), \qquad S(cu) = cS(u)$$

    Let $w_1 = S(u)$, $w_2 = S(v)$, $w_3 = S(u + v)$. Then the statement above is equivalent to saying that we need to show that

    $$w_1 + w_2 = w_3$$

    From the equations above, $T(w_1) = u$, $T(w_2) = v$ and $T(w_3) = u + v$. By the linearity of T,

    $$T(w_1 + w_2) = T(w_1) + T(w_2) = T(w_3)$$

    Since T is invertible,

    $$S(T(w_1 + w_2)) = S(T(w_3)) \Rightarrow w_1 + w_2 = w_3$$

    Side remark: If a function f is not invertible, then $f(x) = f(y)$ does NOT necessarily imply that $x = y$.

    With the same setup let $w_1 = S(cu)$, and $w_2 = cS(u)$. We show that $w_1 = w_2$:

    $$T(w_1) = cu \quad \text{and} \quad T\left(\frac{1}{c}w_2\right) = u \Rightarrow T(w_1) = c \cdot T\left(\frac{1}{c}w_2\right) = T(w_2)$$

    by the linearity of T. Therefore, we have shown that $T(w_1) = T(w_2)$. Since T is invertible, apply S to both sides so that $w_1 = w_2$.