Eigenvalues and Eigenvectors

These notes take portions of sections 5.1, 5.2 and 5.5. We'll look at 5.3 and 5.4 later.

Definition:

Let A be $n \times n$. If there is a constant λ and a non-zero vector $\mathbf{x}$ so that

$$A\mathbf{x} = \lambda\mathbf{x}$$

then λ is said to be an eigenvalue of A, and $\mathbf{x}$ is an associated eigenvector.

For the notation that follows, note that $\lambda\mathbf{x} = \lambda I_n\mathbf{x}$. In what follows, we will learn how to compute λ and $\mathbf{x}$- First some algebra:

$$A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0} \quad \Leftrightarrow \quad (A - \lambda I_n)\,\mathbf{x} = \mathbf{0}$$

Think of this as a matrix equation $B\mathbf{x} = \mathbf{0}$. We know (from the Invertible Matrix Theorem) that if B is invertible, then this equation has only one solution- the trivial solution $\mathbf{x} = \mathbf{0}$. Since we require a non-trivial solution (in our definition above, we said that $\mathbf{x} \neq \mathbf{0}$), that means that B must NOT be invertible. By the IMT, we can use the determinant- that is, we require that $\det(B) = 0$, or:

For $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$ to have a non-trivial solution $\mathbf{x}$, we require that λ be first determined so that

$$\det(A) = 0 \quad \text{or equivalent notation} \quad |A - \lambda I| = 0$$

Important side note:

Definition: The equation $|A - \lambda I| = 0$ is called the **characteristic equation** for the matrix A. It will be shown that the characteristic equation is always a polynomial in λ, so that $p(\lambda) = |A - \lambda I|$ is called the **characteristic polynomial** for the (square) matrix A.

Now continuing:

Once we have found that λ, we go back and solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$ for $\mathbf{x}$.

Example:

Find the eigenvectors and eigenvalues of the matrix A:

$$A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$$

SOLUTION:

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda) + 2 = 0 \quad \Rightarrow \quad \lambda^2 - 5\lambda + 6 = 0$$

Therefore, $\lambda = 3$ and $\lambda = 2$. Now, for each λ, find the eigenvectors:

- For the eigenvalue $\lambda = 3$

$$(A - \lambda I)\mathbf{x} = \mathbf{0} \quad \Rightarrow \quad \left[\begin{array}{rr|r} 4-3 & -2 & 0 \\ 1 & 1-3 & 0 \end{array}\right] \quad \Rightarrow \quad \left[\begin{array}{rr|r} 1 & -2 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

  Put this answer in parametric vector form. Notice that the matrix is rank one (we chose λ that way- this can be used to check your answer)

$$\begin{array}{rl} x_1 & = 2x_2 \\ x_2 & = x_2 \end{array} \quad \Rightarrow \quad \mathbf{x} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

  Notice that any constant multiple of $[2, 1]^T$ can be used as an eigenvector.

- For the eigenvalue $\lambda = 2$, we repeat the process:

$$(A - \lambda I)\mathbf{x} = \mathbf{0} \quad \Rightarrow \quad \left[\begin{array}{rr|r} 4-2 & -2 & 0 \\ 1 & 1-2 & 0 \end{array}\right] \quad \Rightarrow \quad \left[\begin{array}{rr|r} 2 & -2 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$\begin{array}{rl} x_1 & = x_2 \\ x_2 & = x_2 \end{array} \quad \Rightarrow \quad \mathbf{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For completeness, we can verify that $[2, 1]^T$ and $[1, 1]^T$ are eigenvectors of A:

$$A\mathbf{x} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8-2 \\ 2+1 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and

$$A\mathbf{x} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4-2 \\ 1+1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Eigenspaces

We have seen that, if λ is an eigenvalue for A and $\mathbf{x}$ is an eigenvector, then so is $c\mathbf{x}$ for any constant c (in fact, the constant may in fact be either real or complex). Generally speaking, for a given λ, if we find k linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$, then any linear combination of them is also an eigenvector- The span of the eigenvectors is a special subspace called the **eigenspace**, and is denoted by E_λ.

We'll prove this in class, but you might note that if A is a 2×2 matrix, then the corresponding eigenspace is either 1 or 2 dimensional, and so it would be spanned by one or two linearly independepent eigenvectors.

EXAMPLE: Find the eigenvalues and eigenspace associated with the identity matrix.

More examples: Complex and repeated eigenvalues

It is also possible that there is a double eigenvalue. For example,

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

has $\lambda = 1$ as a double root to the characteristic equation. However, unlike the previous example using the identity matrix (which also had a double root $\lambda = 1$), this time there is only one eigenvector for the eigenspace.

Definition: The **algebraic multiplicity** of an eigenvalue is the number of times λ is repeated as a root to the characteristic equation. The **geometric multiplicity** is the dimension of the eigenspace. If these two numbers are not equal, we say that the matrix A is *defective*.

Example

Find the eigenvalues and the algebraic and geometric multiplicities of each, if

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

SOLUTION: The matrix is upper triangular, so the eigenvalues are $\lambda = 1, 1, 3$. Therefore, $\lambda = 1$ has an algebraic multiplicity of 2 and $\lambda = 3$ has an algebraic multiplicity of 1. Finding the eigenspaces, we find that λ has an geometric multiplicity of 1 and so does $\lambda = 3$ (so A is defective).

Example

Find the eigenvalues (and eigenspaces) for

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

SOLUTION:

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 4 = 0 \quad \Rightarrow \quad \lambda^2 - 2\lambda + 5 = 0$$

The quadratic formula can be used to get $\lambda = 1 + 2i$ and $\lambda = 1 - 2i$.

For $\lambda = 1 - 2i$, the matrix $A - \lambda I$ becomes:

$$\begin{bmatrix} 2i & 2 \\ -2 & 2i \end{bmatrix} \quad \Rightarrow \quad 2ix_1 + x_2 = 0 \quad \Rightarrow \quad \mathbf{x} = x_1 \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

As a representative of this space, we take $\mathbf{x} = [1, -i]^T$. In the case of complex eigenvalues, there are some things we should note:

- If you multiply $\mathbf{x}$ by any *complex number*, you still get an eigenvector. (Before, we only used real numbers for the constant).
- If $\mathbf{x}$ is the eigenvector for λ, then the complex conjugate of $\mathbf{x}$ is the eigenvector for the complex conjugate of λ. In this case,

  $$\lambda_2 = 1 + 2i \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

  You should verify that directly by comparing $A\mathbf{x}$ and $\lambda\mathbf{x}$.

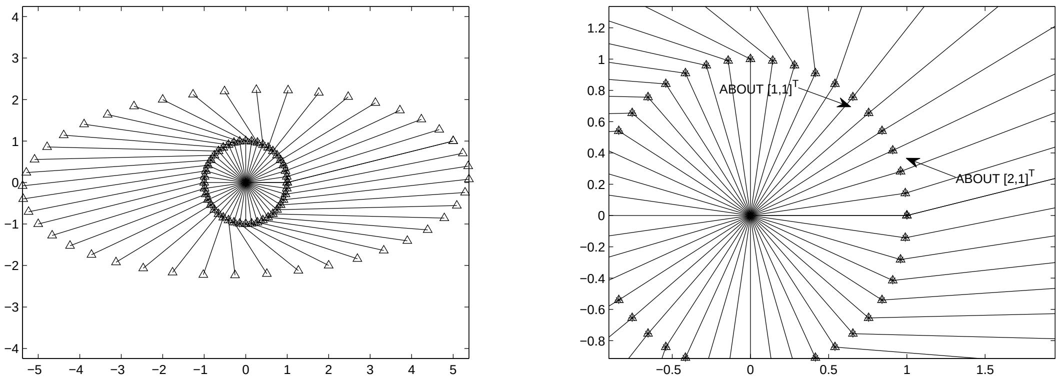


Figure 1: An "eigenpicture" showing the action of a matrix A on the unit vectors around the unit circle. At the end of each unit vector, we plot $A\mathbf{x}$. If $\mathbf{x}$ is an eigenvector of A, the vectors should be parallel.

Visualizing the eigenvectors

It is possible to visualize eigenvectors in two dimensions by plotting a vector $\mathbf{x}$ (usually chosen around the unit circle), then plot $A\mathbf{x}$ starting at $\mathbf{x}$. If $\mathbf{x}$ is an eigenvector of A, then the vectors should be parallel. In this last case, Figure 1 is the "eigenpicture"[1] for the matrix in the last example (together with a zoom into the unit circle).

For the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}$$

the eigenvalues are $\pm\sqrt{10}$. The eigenvectors won't simplify too much, but in Figure 2, we can estimate them. We also see what happens with a negative eigenvalue- The vectors go approximately in the opposite direction of the vector. Zooming in, we can approximate the two vectors as shown.

What is the eigenpicture in the case of complex eigenvalues? In Figure 3 we use the matrix from our previous example, and we see that the action of the matrix A is a rotation (and could also include a scaling as well).

Eigenvalues of larger matrices

Since the eigenvalues of an $n \times n$ matrix are the n roots of a polynomial with real coefficients, by the Fundamental Theorem of Algebra, we know there are exactly n solutions (counting the multiplicity, and allowing for complex solutions).

For the 2×2 matrix, the eigenvalues are either: Two real and distinct values, one real value (repeated twice), or a complex conjugate pair (of the form $\lambda = a \pm bi$).

For the 3×3 matrix, the characteristic polynomial would be a polynomial of degree three. The choice of roots would be: All real (three distinct reals, or two distinct with one repeating, or one triple root), or one real root (multiplicity one) with 2 complex conjugate roots.

And so on.

[1] "Eigenpictures", College Math. Journal, 1995

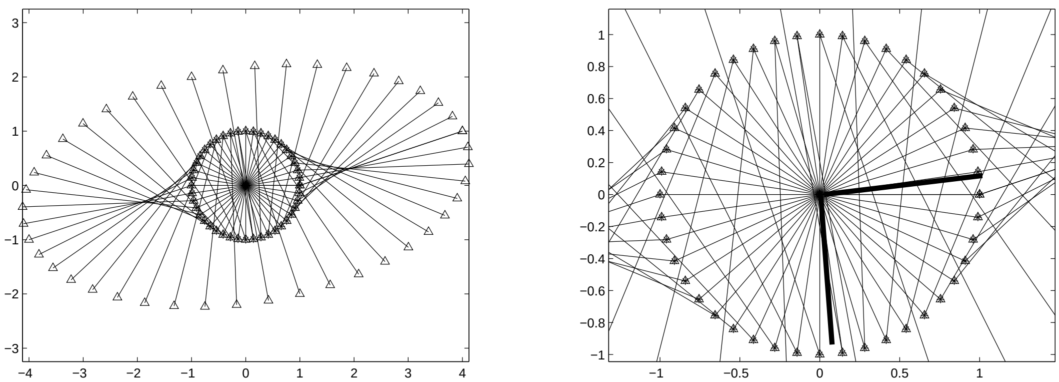


Figure 2: An “eigenpicture” for the second example. In this case, the zoom to the right shows the approximate location of the eigenvectors- The vector going to the right corresponds to a positive eigenvalue. The vector going down corresponds to a negative eigenvalue.

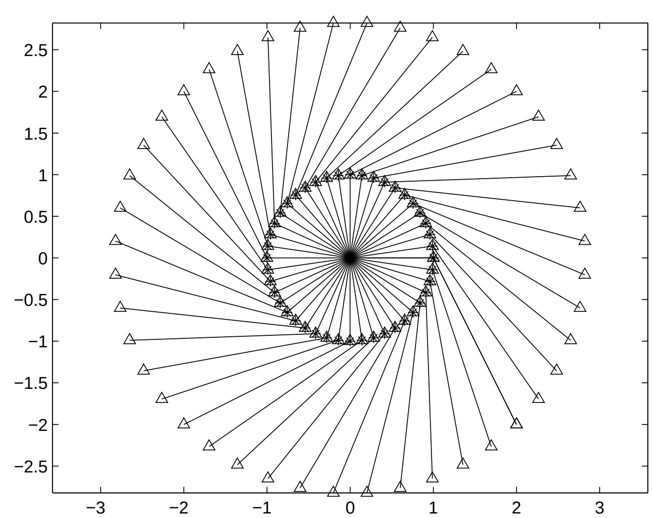


Figure 3: An “eigenpicture” for complex eigenvalues.

Exercises

NOTE: The material here (with the exception of the eigenpictures) are generally taken from sections 5.1, 5.2 and 5.5. We'll be backing into 5.3 and 5.4 later.

1. Pg. 308, 2, 5, 9, 13, 17, 18.

2. Pg. 317, 1, 3, 9, 11, 15.

3. Pg. 341, 1, 3.

4. For each matrix below, find the eigenvalues and the corresponding eigenspace.

$$A_1 = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$

   Hint for A_4: $\lambda = 1$ is a double root of the characteristic equation.

5. Prove Theorem 2, p. 307. Yes, the proof is given in the text, but go through it carefully and be able to provide justification for each step.

6. Show that if λ is an eigenvalue of a matrix A, and we find two linearly independent eigenvectors for λ, call them $\mathbf{u}$ and $\mathbf{v}$, then any linear combination of $\mathbf{u}$ and $\mathbf{v}$ is also an eigenvector corresponding to λ.

7. Show that $\lambda = 0$ is an eigenvalue for matrix A if and only if A is not invertible.

8. Show that if λ is an eigenvalue of the invertible matrix A, then $1/\lambda$ is an eigenvalue of A^{-1}.

9. Show that if λ is an eigenvalue of matrix A, then λ is an eigenvalue of A^T.

10. Show that if λ is an eigenvalue of matrix A, then λ^n is an eigenvalue of A^n.

11. Define the following:

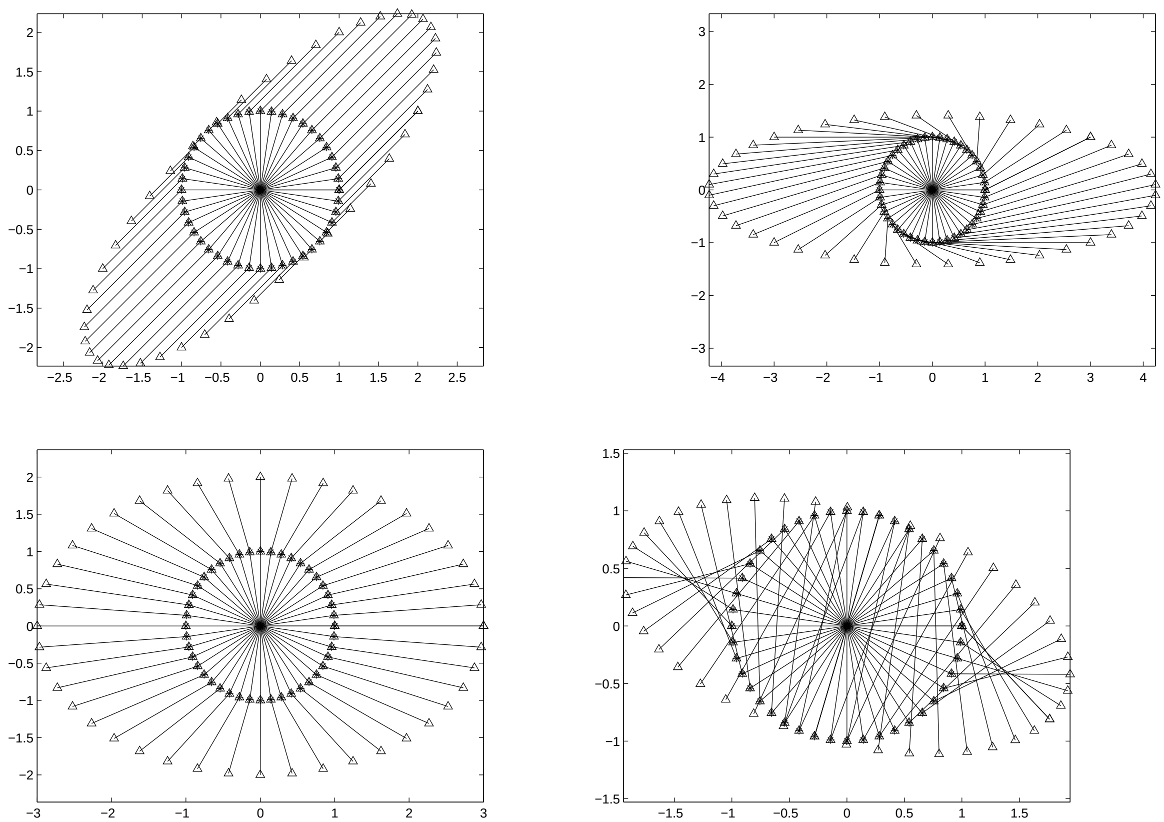
$$A = \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix} \qquad S_1 = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \qquad S_2 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

   (a) Find the eigenvalues of A.
   (b) Compute $S_1AS_1^{-1}$, then find the eigenvalues of $S_1AS_1^{-1}$.
   (c) Repeat using S_2: Compute the eigenvalues of SAS^{-1}.
   (d) Make a conjecture about the eigenvalues of SAS^{-1}, then prove it.

12. Given the matrix A below, find one eigenvalue with no calculation. Justify your answer.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

13. For each of the "eigenpictures" below, estimate the eigenvectors and eigenvalues if they are real. Otherwise, state that they are complex.


2
1.5
1
0.5
0
−0.5
−1
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−2.5 −2 −1.5 −1 −0.5 0 0.5 1 1.5 2 2.5
3
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