Linear Algebra- Final Exam Review

New material since Exam 3:

1. Definitions: Gram-Schmidt, QR, least squares solution, normal equations,
2. The Spectral Theorem is the central theorem of 7.1, and the SVD is the central theorem of 7.4 (know what the factorization produces), and the connection of the SVD to the 4 fundamental subspaces.

For earlier material, look over the concept reviews and review questions from the previous exams. You should also be sure and look over your old exams and homework.

Some "random" questions:

1. Let A be invertible. Show that, if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent vectors, so are $A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3$. NOTE: It should be clear from your answer that you know the definition.
2. Find the line of best first for the data:

x	0	1	2	3
y	1	1	2	2

3. Let $A = \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}$. (a) Is A orthogonally diagonalizable? If so, orthogonally diagonalize it! (b) Find the SVD of A.
4. Let V be the vector space spanned by the functions:

$$f_1(x) = x\sin(x) \qquad f_2(x) = x\cos(x) \qquad f_3(x) = \sin(x) \qquad f_4(x) = \cos(x)$$

   Define the operator $D : V \to D$ as the derivative.

   (a) Find the matrix A of the operator D relative to the basis f_1, f_2, f_3, f_4
   (b) Find the eigenvalues of A.
   (c) Is the matrix A diagonalizable?

5. Short answer:

   (a) Let H be the subset of vectors in $\mathbb{R}^3$ consisting of those vectors whose first element is the sum of the second and third elements. Is H a subspace?
   (b) Explain why the image of a linear transformation $T : V \to W$ is a subspace of W
   (c) Is the following matrix diagonalizable? Explain. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 8 \\ 0 & 0 & 13 \end{bmatrix}$

(d) If the column space of an 8×4 matrix A is 3 dimensional, give the dimensions of the other three fundamental subspaces. Given these numbers, is it possible that the mapping $\mathbf{x} \to A\mathbf{x}$ is one to one? onto?

6. Find a basis for the null space, row space and column space of A, if $A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 5 & 5 \\ 0 & 0 & 3 & 3 \end{bmatrix}$

7. Find an orthonormal basis for $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ using Gram-Schmidt (you might wait until the very end to normalize all vectors at once):

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

8. Find the QR factorization of the matrix $A[\mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_3]$, defined in the previous problem.

9. Given the QR factorization of a matrix A, show that the least squares solution to $A\mathbf{x} = \mathbf{b}$ is $\hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$. (Hint: Start with $A\hat{\mathbf{x}}$, and see where you go).

10. Using the QR factorization in the previous problem for matrix A, find the least squares solution to $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = [1, 0, 0, 0]^T$.

11. Let $\mathbb{P}_n$ be the vector space of polynomials of degree n or less. Let W_1 be the subset of $\mathbb{P}_n$ consisting of $\mathbf{p}(t)$ so that $\mathbf{p}(0)\mathbf{p}(1) = 0$. Let W_2 be the subset of $\mathbb{P}_n$ consisting of $\mathbf{p}(t)$ so that $\mathbf{p}(2) = 0$. Which if the two is a subspace of $\mathbb{P}_n$?

12. For each of the following matrices, find the characteristic equation, the eigenvalues and a basis for each eigenspace:

$$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

13. Define $T : P_2 \to \mathbb{R}^3$ by: $T(p) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$

(a) Find the image under T of $p(t) = 5 + 3t$.

(b) Show that T is a linear transformation.

(c) Find the kernel of T. Does your answer imply that T is $1-1$? Onto? (Review the meaning of these words: kernel, one-to-one, onto)

(d) Find the matrix for T relative to the basis $\{1, t, t^2\}$ for P_2. (This means that the matrix will act on the *coordinates* of p).

14. Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$ so that $\|\mathbf{v}\| = 1$, and let $Q = I - 2\mathbf{v}\mathbf{v}^T$. Show (by direct computation) that $Q^2 = I$.

15. Let A be $m \times n$ and suppose there is a matrix C so that $AC = I_m$. Show that the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$. Hint: Consider $AC\mathbf{b}$.

16. If B has linearly dependent columns, show that AB has linearly dependent columns. Hint: Consider the null space.

17. If λ is an eigenvalue of A, then show that it is an eigenvalue of A^T.

18. Let $\boldsymbol{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, Let S be the parallelogram with vertices at $\mathbf{0}, \boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{u} + v$. Compute the area of S.

19. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $B = \begin{bmatrix} a+2g & b+2h & c+2i \\ d+3g & e+3h & f+3i \\ g & h & i \end{bmatrix}$, and $C = \begin{bmatrix} g & h & i \\ 2d & 2e & 2f \\ a & b & c \end{bmatrix}$.

    If $\det(A) = 5$, find $\det(B), \det(C), \det(BC)$.

20. Let $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix} \begin{bmatrix} 4 \\ -6 \end{bmatrix} \right\}$, and $\mathcal{C} = \left\{ \begin{bmatrix} 4 \\ 5 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix} \right\}$. Write down the matrices that take $[x]_C$ to $[x]_B$ and from $[x]_B$ to $[x]_C$.

21. Define an *isomorphism:*

22. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -8 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \end{bmatrix} \right\}$$

    Find at least two $\mathcal{B}-$coordinate vectors for $\mathbf{x} = [1, 1]^T$.

23. Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for vector space V. Explain why the $\mathcal{B}-$coordinate vectors of $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ are the columns of the $n \times n$ identity matrix:

24. Find the volume of the parallelepiped formed by $\mathbf{0}$, $\mathbf{a}$,$\mathbf{b}$, $\mathbf{c}$, $\mathbf{a}+\mathbf{b}$, $\mathbf{c}+\mathbf{b}$, $\mathbf{c}+\mathbf{a}$, and the sum of all three.

$$\mathbf{a} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$