Linear Algebra- Final Exam Review

1. Let A be invertible. Show that, if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent vectors, so are $A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3$. NOTE: It should be clear from your answer that you know the definition.

   SOLUTION: We need to show that the only solution to:

   $$c_1 A\mathbf{v}_1 + c_2 A\mathbf{v}_2 + c_3 A\mathbf{v}_3 = 0$$

   is the trivial solution. Factoring out the matrix A,

   $$A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = 0$$

   Think of the form $A\hat{\mathbf{x}} = \mathbf{0}$. Since A is invertible, the only solution to this is $\hat{\mathbf{x}} = 0$, which implies that the only solution to the equation above is the solution to

   $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = 0$$

   Which is (only) the trivial solution, since the vectors are linearly independent. (NOTE: Notice that if the original vectors had been linearly dependent, this last equation would have non-trivial solutions).

2. Find the line of best first for the data:

x	0	1	2	3
y	1	1	2	2

   Let A be the matrix formed by a column of ones and a column containing the data in x. If we write the original system as

   We form the normal equations $A^TA\boldsymbol{\beta} = A^T\mathbf{y}$-

   $$A^TA = \begin{bmatrix} 14 & 6 \\ 6 & 4 \end{bmatrix} \quad \Rightarrow \quad (A^TA)^{-1} = \frac{1}{20}\begin{bmatrix} 4 & -6 \\ -6 & 14 \end{bmatrix}$$

   The solution is $\hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{y} = \frac{1}{10}[4, 9]^T$

3. Let $A = \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}$. (a) Is A orthogonally diagonalizable? If so, orthogonally diagonalize it! (b) Find the SVD of A.

   SOLUTION: Yes it is (it is symmetric). The eigenvalues are the diagonal elements, $\lambda = -3, 0$. The eigenvectors are $[1, 0]^T$ and $[0, 1]^T$, respectively, so $P = I$.

   Because one eigenvalue is negative, the SVD is slightly different. The singular values are
   $sigma = \sqrt{9} = 3$ and 0. We can put the negative sign with the matrix V to compensate (so that the factorization works out).

   NOTE: The SVD of a matrix, like the eigenvector decomposition of a matrix, need not be unique. Any representatives from the eigenspaces will make the factorization work for the diagonalization, and in the SVD, $\pm$ signs are chosen to be sure the diagonal entries of Σ are positive.

4. Let V be the vector space spanned by the functions:

$$f_1(x) = x\sin(x) \qquad f_2(x) = x\cos(x) \qquad f_3(x) = \sin(x) \qquad f_4(x) = \cos(x)$$

Define the operator $D : V \to V$ as the derivative.

(a) Find the matrix A of the operator D relative to the basis f_1, f_2, f_3, f_4

SOLUTION:

$$Df_1 = \sin(x) + x\cos(x) = f_3 + f_2$$

This implies that in coordinates,

$$D\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Similarly, we should find that:

$$D\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad D\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad D\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

The matrix for D is the matrix formed from the columns above (in order).

(b) Find the eigenvalues of A.

SOLUTION: Taking the determinant of $A - \lambda I$, we find that the characteristic polynomial is (I used the third column):

$$(\lambda^2 + 1)^2 = 0$$

so $\lambda = \pm i, \pm i$.

(c) Is the matrix A diagonalizable?

SOLUTION: No, not using real numbers.

5. Short answer:

(a) Let H be the subset of vectors in $\mathbb{R}^3$ consisting of those vectors whose first element is the sum of the second and third elements. Is H a subspace?

SOLUTION: One way of showing that a subset is a subspace is to show that the subspace can be represented by the span of some set of vectors. In this case,

$$\begin{bmatrix} a+b \\ a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Because H is the span of the given vectors, it is a subspace.

(b) Explain why the image of a linear transformation $T : V \to W$ is a subspace of W

SOLUTION: Maybe "Prove" would have been better than "Explain", since we want to go through the three parts:

i. $0 \in T(V)$ since $0 \in V$ and $T(0) = 0$.

ii. Let u, v be in $T(V)$. Then there is an x, y in V so that $T(x) = u$ and $T(y) = v$. Since V is a subspace, $x+y \in V$, and therefore $T(x+y) = T(x)+T(y) = u+v$ so that $u + v \in T(V)$.

iii. Let $u \in T(V)$. Show that $cu \in T(V)$ for all scalars c. If $u \in T(V)$, there is an x in V so that $T(x) = u$. Since V is a subspace, $cu \in V$, and $T(cu) \in T(V)$. By linearity, this means $cT(u) \in T(V)$.

(OK, that probably should not have been in the short answer section)

(c) Is the following matrix diagonalizable? Explain. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 8 \\ 0 & 0 & 13 \end{bmatrix}$

SOLUTION: Yes. The eigenvalues are all distinct, so the corresponding eigenvectors are linearly independent.

(d) If the column space of an 8×4 matrix A is 3 dimensional, give the dimensions of the other three fundamental subspaces. Given these numbers, is it possible that the mapping $\mathbf{x} \to A\mathbf{x}$ is one to one? onto?

SOLUTION: If the column space is 3-d, so is the row space. Therefore the null space (as a subspace of $\mathbb{R}^4$) is 1 dimensional and the null space of A^T is 5 dimensional. Since the null space has more than the zero vector, $A\mathbf{x} = \mathbf{0}$ has non-trivial solutions, so the matrix mapping will not be 1-1. Since the column space is a three dimensional subspace of $\mathbb{R}^8$, the mapping cannot be onto.

6. Find a basis for the null space, row space and column space of A, if $A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 5 & 5 \\ 0 & 0 & 3 & 3 \end{bmatrix}$

The basis for the column space is the set containing the first and third columns of A. A basis for the row space is the set of vectors $[1, 1, 0, 0]^T, [0, 0, 1, 1]^T$. A basis for the null space of A is $[-1, 1, 0, 0]^T, [0, 0, -1, 1]^T$.

7. Find an orthonormal basis for $W = \text{Span}\,\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ using Gram-Schmidt (you might wait until the very end to normalize all vectors at once):

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

Using Gram Schmidt (before normalization), we get

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

8. Find the QR factorization of the matrix $A[\mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_3]$, defined in the previous problem.

   SOLUTION: Gets a bit messy by hand- We can write it symbolically as $R = Q^T A$, where the columns of Q are the normalized vectors from the previous problem.

9. Given the QR factorization of a matrix A, show that the least squares solution to $A\mathbf{x} = \mathbf{b}$ is $\hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$. (Hint: Start with $A\hat{\mathbf{x}}$, and see where you go).

   SOLUTION: Using the hint, and substitute the expression:

$$A\hat{\mathbf{x}} = (QR)R^{-1}Q^T\mathbf{b} = QQ^T\mathbf{b}$$

   which is the orthogonal projection of $\mathbf{b}$ into the column space of Q, which was the column space of A. Therefore, we obtain $\hat{\mathbf{b}}$; the element of the column space closest to $\mathbf{b}$.

10. Using the QR factorization in the previous problem for matrix A, find the least squares solution to $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = [1, 0, 0, 0]^T$.

    SOLUTION: As in the text (6.4), we see that even small problems can be messy- Try finding the solution in Matlab (sorry- I thought it would work out a little cleaner by hand).

11. Let $\mathbb{P}_n$ be the vector space of polynomials of degree n or less. Let W_1 be the subset of $\mathbb{P}_n$ consisting of $\mathbf{p}(t)$ so that $\mathbf{p}(0)\mathbf{p}(1) = 0$. Let W_2 be the subset of $\mathbb{P}_n$ consisting of $\mathbf{p}(t)$ so that $\mathbf{p}(2) = 0$. Which of the two is a subspace of $\mathbb{P}_n$?

    SOLUTION: First consider W_1. The zero polynomial is in W_1. The condition states that either $p(0)$ or $p(1)$ must be zero, but not both- That could cause a problem. In fact, what happens with a sum? Let $f(t) = p(t) + q(t)$. Then

$$f(0)f(1) = (p(0) + q(0))(p(1) + q(1)) = p(0)p(1) + p(0)q(1) + p(1)q(0) + q(0)q(1)$$

    The middle terms will cause a problem. For example, how about $p(0) = 4$, $p(1) = 0$ and $q(0) = 0$, but $q(1) = 3$? They each satisfy the condition for W_1, but the sum will not. Therefore, W_1 is not a subspace of $\mathbb{P}_n$.

    For W_2, we will have a subspace (check the three conditions).

12. For each of the following matrices, find the characteristic equation, the eigenvalues and a basis for each eigenspace:

$$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

    SOLUTION: For matrix A, $\lambda = 3, 5$. Eigenvectors are $[1, 1]^T$ and $[2, 1]^T$, respectively.

    For matrix B, for $\lambda = 3+i$, an eigenvector is $[1, i]^T$. The other eigenvalue and eigenvector are the complex conjugates.

    For matrix C, expand along the 2d row. $\lambda = 2$ is a double eigenvalue with eigenvectors $[0, 1, 0]^T$ and $[1, 0, 1]^T$. The third eigenvalue is $\lambda = 0$ with eigenvector $[-1, 0, 1]^T$.

13. Define $T : P_2 \to \mathbb{R}^3$ by: $T(p) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$

(a) Find the image under T of $p(t) = 5 + 3t$.
SOLUTION: $[2, 5, 8]^T$

(b) Show that T is a linear transformation.
SOLUTION: We show it using the definition.

i. Show that $T(p + q) = T(p) + T(q)$:

$$T(p+q) = \begin{bmatrix} p(-1) + q(-1) \\ p(0) + q(0) \\ p(1) + q(1) \end{bmatrix} = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix} + \begin{bmatrix} q(-1) \\ q(0) \\ q(1) \end{bmatrix} = T(p) + T(q)$$

ii. Show that $T(cp) = cT(p)$ for all scalars c.

$$T(cp) = \begin{bmatrix} cp(-1) \\ cp(0) \\ cp(1) \end{bmatrix} = c \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix} = cT(p)$$

(c) Find the kernel of T. Does your answer imply that T is $1 - 1$? Onto? (Review the meaning of these words: kernel, one-to-one, onto)
SOLUTION:
Since the kernel is the set of elements in the domain that map to zero, let's see what what the action of T is on an arbitrary polynomial. An arbitrary vector in P_2 is: $p(t) = at^2 + bt + c$, and:

$$T(at^2 + bt + c) = \begin{bmatrix} a - b + c \\ c \\ a + b + c \end{bmatrix}$$

For this to be the zero vector, $c = 0$. Then $a - b = 0$ and $a + b = 0$, so $a = 0, b = 0$. Therefore, the only vector mapped to zero is the zero vector.
Side Remark: Recall that for any linear function T, if we are solving $T(x) = y$, then the solution can be written as $x = x_p + x_h$, where x_p is the particular solution (it solves $T(x_p) = y$), and $T(x_h) = 0$ (we said x_h is the homogeneous part of the solution). So the equation $T(x) = y$ has at most one solution iff the kernel is only the zero vector (if T was realized as a matrix, we get our familiar setting).

Therefore, T is $1 - 1$. The mapping T will also be onto (see the next part).

(d) Find the matrix for T relative to the basis $\{1, t, t^2\}$ for P_2. (This means that the matrix will act on the *coordinates* of p).
You can verify that the matrix representation for T is:

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

From this we see that the matrix is invertible, so the mapping is both 1-1 and onto (as a mapping from $\mathbb{R}^3$ to $\mathbb{R}^3$).

14. Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$ so that $\|\mathbf{v}\| = 1$, and let $Q = I - 2\mathbf{v}\mathbf{v}^T$. Show (by direct computation) that $Q^2 = I$.

    SOLUTION: This problem is to practice matrix algebra:

    $$Q^2 = (I - 2\mathbf{v}\mathbf{v}^T)(I - 2\mathbf{v}\mathbf{v}^T) = I^2 - 2I\mathbf{v}\mathbf{v}^T - 2\mathbf{v}\mathbf{v}^T I + 4\mathbf{v}\mathbf{v}^T\mathbf{v}\mathbf{v}^T = I - 4\mathbf{v}\mathbf{v}^T + 4\mathbf{v}(1)\mathbf{v}^T = I$$

15. Let A be $m \times n$ and suppose there is a matrix C so that $AC = I_m$. Show that the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$. Hint: Consider $AC\mathbf{b}$.

    SOLUTION: Using the hint, we see that $AC\mathbf{b} = \mathbf{b}$. Therefore, given an arbitrary vector $\mathbf{b}$, the solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = C\mathbf{b}$.

16. If B has linearly dependent columns, show that AB has linearly dependent columns. Hint: Consider the null space.

    SOLUTION: If B has linearly dependent columns, then the equation $B\mathbf{x} = \mathbf{0}$ has non-trivial solutions. Therefore, the equation $AB\mathbf{x} = \mathbf{0}$ has (the same) non-trivial solutions, and the columns of AB must be linearly dependent.

17. If λ is an eigenvalue of A, then show that it is an eigenvalue of A^T.

    SOLUTION: Use the properties of determinants. Given

    $$|A - \lambda I| = |(A - \lambda I)^T| = |A^T - \lambda I^T| = |A^T - \lambda I|$$

    the solutions to $|A - \lambda I| = 0$ and $|A^T - \lambda I| = 0$ are exactly the same.

18. Let $\boldsymbol{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, Let S be the parallelogram with vertices at $\mathbf{0}, \boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{u} + v$. Compute the area of S.

    SOLUTION: The area of the parallelogram formed by two vectors in $\mathbb{R}^2$ is the determinant of the matrix whose columns are those vectors. In this case, that would be 4.

19. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $B = \begin{bmatrix} a+2g & b+2h & c+2i \\ d+3g & e+3h & f+3i \\ g & h & i \end{bmatrix}$, and $C = \begin{bmatrix} g & h & i \\ 2d & 2e & 2f \\ a & b & c \end{bmatrix}$.

    If $\det(A) = 5$, find $\det(B), \det(C), \det(BC)$.

    SOLUTION: This question reviews the relationship between the determinant and row operations. The determinant of B is 5. The determinant of C is 10. The determinant of BC is 50.

20. Let $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix} \begin{bmatrix} 4 \\ -6 \end{bmatrix} \right\}$, and $\mathcal{C} = \left\{ \begin{bmatrix} 4 \\ 5 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix} \right\}$. Write down the matrices that take $[x]_C$ to $[x]_B$ and from $[x]_B$ to $[x]_C$.

SOLUTION: Since $\mathbf{x} = P_B[\mathbf{x}]_B$ and $\mathbf{x} = P_C[\mathbf{x}]_C$, then

$$[\mathbf{x}]_C = P_C^{-1}P_B[\mathbf{x}]_B \qquad [\mathbf{x}]_B = P_B^{-1}P_C[\mathbf{x}]_C$$

Doing the calculations,

$$P_C^{-1}P_B = \begin{bmatrix} -1/2 & 5 \\ -1 & 5 \end{bmatrix} \qquad P_B^{-1}P_C = \begin{bmatrix} 2 & 2 \\ 2/5 & 1/5 \end{bmatrix}$$

21. Define an *isomorphism:* A one-to-one linear transformation between vector spaces (see p. 251)

22. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -8 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \end{bmatrix} \right\}$$

Find at least two $\mathcal{B}-$coordinate vectors for $\mathbf{x} = [1,1]^T$.

SOLUTION: The null space of the matrix is spanned by $[5,-1,1]^T$ (found by row reduction. The particular part of the solution is $[5,-2,0]^T$. So we can find an infinite number of solutions.

23. Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for vector space V. Explain why the $\mathcal{B}-$coordinate vectors of $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ are the columns of the $n \times n$ identity matrix:

SOLUTION: $[\mathbf{b}_1]_B = [1,0,0,\ldots,0]^T$ because $\mathbf{b}_1 = 1\mathbf{b}_1 + 0\mathbf{b}_2 + \cdots + 0\mathbf{b}_n$. A similar argument shows that $[\mathbf{b}_i]_B = \vec{e}_i$.

24. Find the volume of the parallelepiped formed by $\mathbf{0}$, $\mathbf{a}$,$\mathbf{b}$, $\mathbf{c}$, $\mathbf{a}+\mathbf{b}$, $\mathbf{c}+\mathbf{b}$, $\mathbf{c}+\mathbf{a}$, and the sum of all three.

$$\mathbf{a} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The volume is the determinant. In this case, expand along the second row, and get 3.