Topic List: Exam 2, Math 300

Matrix Algebra and Inverses (2.1-2.3)

1. Matrix multiplication can be viewed in multiple ways:
   - As the standard matrix for function composition.
   - Useful for theoretical work, this is our definition: If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$, where the product is defined column-wise as:
   $$AB = A[\mathbf{b}_1 \cdots \mathbf{b}_p] = [A\mathbf{b}_1 \quad \cdots \quad A\mathbf{b}_p]$$
   - The "row-column" rule for actual computation.
   - "The columns of AB are linear combinations of the columns of A".
2. Other matrix operations: Addition/subtraction, multiply by a scalar, take the transpose of a matrix, raise a matrix to a (positive) integer power. Know the rule for the inverse of a 2×2 matrix. Know the general algorithm for computing an inverse.
3. Rules for transposes and inverses:
   - $(A+B)^T = A^T + B^T$
   - $(AB)^T = B^T A^T$
   - $(AB)^{-1} = B^{-1}A^{-1}$ (if A, B are each invertible)
   - $(A^T)^{-1} = (A^{-1})^T$ if A is invertible.
   - The i^{th} column of A^{-1} is the solution to $A\mathbf{x} = \mathbf{e}_i$.
4. Know the meaning of $1-1$ and onto as it relates to the function $\mathbf{x} \to A\mathbf{x}$
5. The Invertible Matrix Theorem. I will not ask you to simply list these, but I will ask things like: If A is invertible, then discuss the number of pivots A must have. If A is invertible, how many solutions are there to $A\mathbf{x} = \mathbf{0}$?, etc.

Determinants (3.1-3.3)

List of things

Be able to compute determinants using a cofactor expansion along any row or column. Compute a determinant for upper or lower triangular matrix. Be able to compute a determinant by first performing row reduction. Cramer's Rule (general case and for 2×2 systems), volume of a parallelepiped, area of a parallelogram. Area after a linear transformation T. A is invertible only if $\det(A) \neq 0$.

Properties

For the following, assume A, B are square matrices. For the last item, assume A is invertible.

- If A is $n \times n$, then $\det(kA) = k^n \det(A)$.
- $\det(AB) = \det(A)\det(B)$
- $\det(A^T) = \det(A)$
- $\det(A^{-1}) = 1/\det(A)$

Vector Spaces, (4.1-4.3)

- You don't need to memorize the 10 axioms on page 217.
- Be familiar with some template vector spaces:

$$\mathbb{R}^n, \mathbb{P}_n(t), C[a, b], M_{m\times n}$$

- Know the definition of a linearly independent set, a subspace, and a basis.
- Know the theorem that states that a spanning set always forms a subspace.
- Show that something is or is not a subspace.
- The 4 fundamental subspaces for a matrix A $(m \times n)$ and their definitions:
  - The Col(A) is the span of the columns of A (and is a subspace of $\mathbb{R}^n$). Note: Row operations do not affect the relationship among the columns of A.
  - The Row(A) is the span of the rows of A (and is a subspace of $\mathbb{R}^n$.) Note: Row operations DO affect the relationship among the rows of A.
  - The Null(A) is the null space of A, and is the set of all $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$. It is a subspace of $\mathbb{R}^n$.
  - The Null(A^T) is the null space of A^T, and is similarly defined as the set of all solutions to $A^T\mathbf{b} = \mathbf{0}$. It is a subspace of $\mathbb{R}^m$.
- Know the definition of a basis.
- Be able to compute a basis for the first three fundamental subspaces of a matrix (no need to compute the null space of A^T).
- Know the definition of a kernel, and contrast with the null space.