MATH 300, Second Exam REVIEW QUESTIONS

NOTE: You may use a calculator for this exam- You only need something that will perform basic arithmetic.

1. Let $\boldsymbol{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, Let S be the parallelegram with vertices at $\mathbf{0}, \boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{u} + v$. Compute the area of S.

2. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $B = \begin{bmatrix} a+2g & b+2h & c+2i \\ d+3g & e+3h & f+3i \\ g & h & i \end{bmatrix}$, and $C = \begin{bmatrix} g & h & i \\ 2d & 2e & 2f \\ a & b & c \end{bmatrix}$.
   If $\det(A) = 5$, find $\det(B), \det(C), \det(BC)$.

3. Assume that A and B are row equivalent, where:
$$A = \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ -2 & -3 & 1 & -1 & -5 \\ -3 & -4 & 0 & -2 & -3 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 4 & 0 & 3 \\ 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
   (a) Find a basis for $\mathrm{Col}(A)$
   (b) Find a basis for $\mathrm{Row}(A)$
   (c) Find a basis for $\mathrm{Null}(A)$

4. Determine if the following sets are subspaces of V. Justify your answers.
$$H = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, a \geq 0, \;\; b \geq 0, \;\; c \geq 0 \right\}, \qquad V = \mathbb{R}^3$$
$$H = \left\{ \begin{bmatrix} a+3b \\ a-b \\ 2a+b \\ 4a \end{bmatrix}, \;\; a, b \text{ in } \mathbb{R} \right\}, \qquad V = \mathbb{R}^4$$
$$H = \{f : f'(x) = f(x)\}, V = C^1[\mathbb{R}]$$
   (C^1 is the space of differentiable functions where the derivative is continuous).
   H is the set of vectors in $\mathbb{R}^3$ whose first entry is the sum of the second and third entries, $V = \mathbb{R}^3$.

5. Prove that, if $T : V \mapsto W$ is a linear transformation between vector spaces V and W, then the range of T is a subspace of W.

6. Let A be an $n \times n$ matrix. Write statements from the Invertible Matrix Theorem that are each equivalent to the statement "A is invertible". Use the following concepts, one in each statement: (a) $\mathrm{Null}(A)$ (b) Basis (c) Determinant

7. If A, B are 4×4 matrices with $\det(A) = 2$ and $\det(B) = -3$, what is the determinant of the following (if you can compute it): (a) $\det(AB)$, (b) $\det(A^{-1})$, (c) $\det(5B)$, (d) $\det(3A - 2B)$, (e) $\det(B^T)$

8. Is it possible that all solutions of a homogeneous system of ten linear equations in twelve variables are multiples of one fixed nonzero solution? Discuss.

9. Show that $\{1, 2t, -2 + 4t^2\}$ is a basis for P_3. (Hint for the span: Show that an arbitrary polynomial $a + bt + ct^2$ can be written as a linear combination of the given vectors.)

10. Let $T : V \to W$ be a linear transformation on vector space V to vector space W. Show that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent vectors in V, then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ are linearly independent vectors in W.

11. (a) The functions $f(t) = t$ and $g(t) = t$ are linearly dependent. Are $f(t) = t$ and $g(t) = -t$ linearly dependent?

(b) Are the functions $f(t) = t$ and $g(t) = |t|$ linearly independent or dependent?

12. Use Cramer's Rule to solve the system:

$$\begin{aligned} 2x_1 + x_2 \quad &= 7 \\ -3x_1 + x_3 \quad &= -8 \\ x_2 + 2x_3 \quad &= -3 \end{aligned}$$

13. Let A be a 3×4 matrix. Construct the elementary matrix E to perform the given row operations, and give the determinant of E:

    (a) $3r_2 + r_1 \to r_1$

    (b) $r_3 \to -3r_3$

    (c) The inverse of the row operation $2r_1 + r_2 \to r_2$.

14. Let A, B be square matrices. Show that if AB is invertible, then so is A. (Hint: If AB is invertible, let W be its inverse so that $ABW = I$.)

15. Let $A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$, and $\mathbf{w} = [2, 1]^T$. Is $\mathbf{w}$ in the column space of A? Is it in the null space of A?

16. Prove directly (by showing the three conditions hold) that the column space of a matrix is a subspace.

17. Let A and B be given below. (a) Find only the first column of AB. (b) Find only the $(3, 1)$ entry of $B^T A$. (c) Determine the inverse of BB^T, if possible.

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}$$

18. Let A be $m \times n$. Suppose that for some matrix C, $CA = I$. Show that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

19. True or False, and give a short reason:

    (a) If $AB = I$, then A is invertible.

    (b) If $BC = BD$, then $C = D$.

    (c) If A, B are $n \times n$, then $(A + B)(A - B) = A^2 - B^2$.

    (d) If $\det(A) = 2$ and $\det(B) = 3$, then $\det(A + B) = 5$.

    (e) Let A be $n \times n$. Then $\det(A^T A) \geq 0$.

    (f) If $A^3 = 0$, then $\det(A) = 0$.