Example Questions, Exam 3, Math 300

1. Let the matrix A and its RREF, R_A, be given as below:

$$A = \begin{bmatrix} 1 & 1 & 7 & 2 & 2 \\ 3 & 0 & 9 & 3 & 4 \\ -3 & 1 & -5 & -2 & 3 \\ 2 & 2 & 14 & 4 & 2 \end{bmatrix} \qquad R_A = \begin{bmatrix} 1 & 0 & 3 & 1 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so that the columns of A are $\mathbf{a}_1, \cdots, \mathbf{a}_5$.

Similarly, define Z and its RREF, R_Z, as:

$$Z = \begin{bmatrix} 4 & 5 & 3 & 4 \\ 5 & 6 & 5 & -3 \\ 10 & -3 & 9 & -106 \\ 4 & 10 & 2 & 44 \end{bmatrix} \qquad R_z = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Label the columns of Z as $\mathbf{z}_1, \cdots, \mathbf{z}_5$.

(a) Find the rank of A and a basis for the column space of A (use the notation $\mathbf{a}_1$, etc.). Similarly, do the same for Z:

(b) You'll notice that the rank of A is the rank of Z. Here is a row reduction using some columns of A and Z:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 4 & 5 & 3 \\ 3 & 0 & 4 & 5 & 6 & 5 \\ -3 & 1 & 3 & 10 & -3 & 9 \\ 2 & 2 & 2 & 4 & 10 & 2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Are the subspaces spanned by the columns of A and Z equal?

(c) Let $\mathcal{B}$ and $\mathcal{C}$ be the sets of basis vectors used for the column spaces of A, Z found in (a). Find the change of basis matrix $P_{\mathcal{B}\leftarrow\mathcal{Z}}$. HINT: Use the row reduction found in part (b).

(d) Find the coordinates of $\mathbf{z}_4$ using the basis vectors in $\mathcal{C}$.

(e) Use the previous two answers to find the coordinates of $\mathbf{z}_4$ in terms of the set $\mathcal{B}$:

2. Let A and its RREF be given as:

$$A = \begin{bmatrix} -1 & -5 & 3 & 9 \\ -48 & -40 & 24 & 92 \\ 94 & 70 & -42 & -166 \\ -48 & -40 & 24 & 92 \end{bmatrix} \qquad R_A = \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & -3/5 & -17/10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We also note two facts: $\lambda = 4$ is an eigenvalue of A, and $\mathbf{u} = [1, 0, 2, 0]^T$ is an eigenvector of A.

(a) Find a basis for the null space of A:

(b) Find a basis for the eigenspace E_4:

(c) What is the eigenvalue for the eigenvector $\mathbf{u}$?

(d) What is the characteristic polynomial of A?

(e) Show that A is diagonalizable by finding an appropriate P and D.

3. Short Answer:

   (a) Show that if A^2 is the zero matrix, the only eigenvalue of A is zero.

   (b) If C is 4×5, what is the largest possible rank of C? What is the smallest possible dimension of the null space of C?

   (c) If A is a 4×7 matrix with rank 3, find the dimensions of the four fundamental subspaces of A.

   (d) Suppose A is 3×3, and $\mathbf{u}$ is an eigenvector of A corresponding to an eigenvalue of 7.

   Is $\mathbf{u}$ an eigenvector of $2I - A$? If so, find the corresponding eigenvalue. If not, explain why not.

   (e) True or false? The columns of $P_{\mathcal{C} \leftarrow \mathcal{B}}$ are linearly independent.

4. Let the subspace W be the span of $\mathbf{u}_1 = [-1, 2, 0]^T$, and $\mathbf{u}_2 = [4, 2, 5]^T$, and let $\mathbf{y} = [9, 7, 17]^T$.

   (a) Find the orthogonal projection of $\mathbf{y}$ into the subspace W.

   (b) Find the distance from $\mathbf{y}$ to the subspace W.

   (c) Decompose $\mathbf{y}$ as a sum of one vector from W and another from $W^\perp$.

   (d) Let the matrix $U = [\mathbf{u}_1 \quad \mathbf{u}_2]$. Is the projection of $\mathbf{y}$ into the subspace W performed by $UU^T\mathbf{y}$? Why or why not (if not, fix U so that it is).

5. Let U be $m \times n$ with orthonormal columns. Show that the length of $U\mathbf{x}$ is the same as the length of $\mathbf{x}$.

6. Show that the eigenvalues of A and A^T are the same.

7. Show that if $\mathbf{u}$, $\mathbf{v}$ are eigenvectors corresponding to distinct eigenvalues, then the vectors are linearly independent.

8. Show that the coordinate mapping (from $n-$dimensional vector space V to $\mathbb{R}^n$) is onto.

9. Find the eigenvalues and eigenspaces for $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$.

10. In 5.5, we're told that if A is 2×2 with $\lambda = a - bi$, then we can write $A = PCP^{-1}$ for a certain P, C. Use this decomposition with $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$ (that is the same matrix as the previous problem).