Example Solutions, Exam 3, Math 300

1. Let the matrix A and its RREF, R_A, be given as below:

$$A = \begin{bmatrix} 1 & 1 & 7 & 2 & 2 \\ 3 & 0 & 9 & 3 & 4 \\ -3 & 1 & -5 & -2 & 3 \\ 2 & 2 & 14 & 4 & 2 \end{bmatrix} \qquad R_A = \begin{bmatrix} 1 & 0 & 3 & 1 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so that the columns of A are $\mathbf{a}_1, \cdots, \mathbf{a}_5$.

Similarly, define Z and its RREF, R_Z, as:

$$Z = \begin{bmatrix} 4 & 5 & 3 & 4 \\ 5 & 6 & 5 & -3 \\ 10 & -3 & 9 & -106 \\ 4 & 10 & 2 & 44 \end{bmatrix} \qquad R_z = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Label the columns of Z as $\mathbf{z}_1, \cdots, \mathbf{z}_5$.

(a) Find the rank of A and a basis for the column space of A (use the notation $\mathbf{a}_1$, etc.). Similarly, do the same for Z:

SOLUTION:

The rank is the dimension of the column space (the number of pivot columns in the RREF), so in this case it is 3. The corresponding columns of A are $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5$ and these form a basis for Col(A).

Similarly, the rank of Z is 3 and a basis for Col(Z) is $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$.

(b) You'll notice that the rank of A is the rank of Z. Here is a row reduction using some columns of A and Z:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 4 & 5 & 3 \\ 3 & 0 & 4 & 5 & 6 & 5 \\ -3 & 1 & 3 & 10 & -3 & 9 \\ 2 & 2 & 2 & 4 & 10 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Are the subspaces spanned by the columns of A and Z equal?

SOLUTION: The RREF tells us that $\mathbf{z}_1, \mathbf{z}_2$ and $\mathbf{z}_3$ can be written in terms of $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_5$. The subspaces are both three dimensional, so if we have any set of 3 linearly independent vectors (in the subspace) we must have a basis (therefore, the basis for Col(Z) could be used for Col(A) and vice-versa).

(c) **Typo: This is the corrected version** (the script Z should be script C) Let $\mathcal{B}$ and $\mathcal{C}$ be the sets of basis vectors used for the column spaces of A, Z found in (a). Find the change of basis matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$. HINT: Use the row reduction found in part (b).

SOLUTION: Use the change of coordinates matrix as defined on p. 273)

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = [[\mathbf{z}_1]_B \;\; [\mathbf{z}_2]_B \;\; [\mathbf{z}_3]_B]$$

And the row reduction gives the three columns as:

$$[\mathbf{z}_1]_B = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \qquad [\mathbf{z}_2]_B = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \qquad [\mathbf{z}_3]_B = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

NOTE that the coordinates are vectors in $\mathbb{R}^3$ because there are three basis vectors in the column space of A. The change in coordinates matrix is square because there are also 3 basis vectors in the column space of Z.

(d) Find the coordinates of $\mathbf{z}_4$ using the basis vectors in $\mathcal{C}$.
SOLUTION: From the initial setup, we see that

$$[\mathbf{z}_4]_C = \begin{bmatrix} -4 \\ 7 \\ -5 \end{bmatrix}$$

(e) Use the previous two answers to find the coordinates of $\mathbf{z}_4$ in terms of the set $\mathcal{B}$:
SOLUTION: Use the change of coordinates matrix:

$$[\mathbf{z}_4]_B = \begin{bmatrix} -1 & 2 & -1 \\ 1 & 3 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -7 \\ -5 \end{bmatrix} = \begin{bmatrix} 23 \\ 27 \\ -18 \end{bmatrix}$$

2. Let A and its RREF be given as:

$$A = \begin{bmatrix} -1 & -5 & 3 & 9 \\ -48 & -40 & 24 & 92 \\ 94 & 70 & -42 & -166 \\ -48 & -40 & 24 & 92 \end{bmatrix} \qquad R_A = \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & -3/5 & -17/10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We also note two facts: $\lambda = 4$ is an eigenvalue of A, and $\mathbf{u} = [1, 0, 2, 0]^T$ is an eigenvector of A.

(a) Find a basis for the null space of A:

SOLUTION: From the RREF, we see that the the null space is two dimensional. Putting the solution to $A\mathbf{x} = \mathbf{0}$ in parametric vector form, we should be able to extract the basis vectors (multiplied by a scalar so that they are integer valued):

$$\left\{ \begin{bmatrix} 0 \\ 3 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 17 \\ 0 \\ 10 \end{bmatrix} \right\}$$

(b) Find a basis for the eigenspace E_4:
(Add the hint: The last column of $B = A - 4I$ is $-2\mathbf{b}_1 - \mathbf{b}_2 - 2\mathbf{b}_3$)
SOLUTION:

This is the null space of $A - 4I$, which we can compute the long way or take the (added) hint. The hint implies that the RREF of $B = A - 4I$ is:

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, the eigenspace is spanned by $[2, 1, 2, 1]^T$.

(c) What is the eigenvalue for the eigenvector $\mathbf{u}$?

Multiply $A\mathbf{u}$ to find that the eigenvalue is 5.

(d) What is the characteristic polynomial of A?

SOLUTION: Create a polynomial with the given zeros:

$$\lambda^2(\lambda - 4)(\lambda - 5)$$

(e) Show that A is diagonalizable by finding an appropriate P and D.

SOLUTION: Set up D and P as usual- we have a full set of linearly independent eigenvectors:

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 5 & 2 & 1 \\ 3 & 17 & 1 & 0 \\ 5 & 0 & 2 & 2 \\ 0 & 10 & 1 & 0 \end{bmatrix}$$

3. Short Answer:

(a) Show that if A^2 is the zero matrix, the only eigenvalue of A is zero.

SOLUTION: We know that if λ is an eigenvalue of A, then λ^2 will be an eigenvalue of A^2. Since A^2 is diagonal with diagonal entries zero, A^2 has a double zero eigenvalue. Therefore, there cannot be a non-zero eigenvalue of A.

(b) If C is 4×5, what is the largest possible rank of C?

SOLUTION: The largest possible rank is 4. (If C has its largest possible rank, it is said to be full rank).

What is the smallest possible dimension of the null space of C?

SOLUTION: The dimension of the null space is the number of free variables in the RREF of the matrix (or the number of non-pivot columns). If the rank is 4, the dimension of the null space must be at 1. (Note: If the rank were smaller than 4, the extra dimensions are added to the null space).

(c) If A is a 4×7 matrix with rank 3, find the dimensions of the four fundamental subspaces of A.

SOLUTION: If A has rank 3, then the dimension of the row space is 3, the dimension of the null space is 4, the dimension of the column space is 3, and the dimension of $\text{Null}(A^T)$ is 1.

(d) Suppose A is 3×3, and $\mathbf{u}$ is an eigenvector of A corresponding to an eigenvalue of 7.

Is $\mathbf{u}$ an eigenvector of $2I - A$? If so, find the corresponding eigenvalue. If not, explain why not.

SOLUTION: Check directly-

$$(2I - A)\mathbf{u} = 2\mathbf{u} - A\mathbf{u} = 2\mathbf{u} - 7\mathbf{u} = -5\mathbf{u}$$

Yes, corresponding to an eigenvalue of -5.

(e) True or false? The columns of $P_{\mathcal{C} \leftarrow \mathcal{B}}$ are linearly independent.

SOLUTION: Yes. They are the coordinates of the basis from set $\mathcal{B}$ in terms of the basis in set $\mathcal{C}$. If they are linearly independent in $\mathcal{B}$, they must be in $\mathcal{C}$ as well (the change of coordinates is linear).

4. Let the subspace W be the span of $\mathbf{u}_1 = [-1, 2, 0]^T$, and $\mathbf{u}_2 = [4, 2, 5]^T$, and let $\mathbf{y} = [9, 7, 17]^T$.

NOTE: Before beginning, you should observe that the vectors given for the subspace W are orthogonal!

(a) Find the orthogonal projection of $\mathbf{y}$ into the subspace W.

SOLUTION: Since $\mathbf{u}_1, \mathbf{u}_2$ are orthogonal, we compute:

$$\hat{\mathbf{y}} = \mathrm{Proj}_W(\mathbf{y}) = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2$$

Which simplifies to $\hat{\mathbf{y}} = \mathbf{u}_1 + 3\mathbf{u}_2 = [11, 8, 15]^T$.

(b) Find the distance from $\mathbf{y}$ to the subspace W.

SOLUTION: The distance is $\|\mathbf{y} - \hat{\mathbf{y}}\| = \sqrt{2^2 + 1^2 + 2^2} = 3$

(c) Decompose $\mathbf{y}$ as a sum of one vector from W and another from $W^{\perp}$.

SOLUTION: The vector in $W^{\perp}$ is $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$, so

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} = \begin{bmatrix} 11 \\ 8 \\ 15 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

(**We might note that** $\mathbf{z}$ is perpendicular to $\mathbf{u}_1, \mathbf{u}_2$, so we know that $\mathbf{z}$ is orthogonal to every vector in W.)

(d) Let the matrix $U = [\mathbf{u}_1 \quad \mathbf{u}_2]$. Is the projection of $\mathbf{y}$ into the subspace W performed by $UU^T\mathbf{y}$? Why or why not (if not, fix U so that it is).

SOLUTION: No- the matrix U must have orthonormal columns. Normalize the first two columns to get that

$$U = \begin{bmatrix} \frac{1}{\sqrt{5}}\mathbf{u}_1 & \frac{1}{3\sqrt{5}}\mathbf{u}_2 \end{bmatrix} = \frac{1}{3\sqrt{5}}[3\mathbf{u}_1 \quad \mathbf{u}_2] = \frac{1}{3\sqrt{5}} \begin{bmatrix} -3 & 4 \\ 6 & 2 \\ 0 & 5 \end{bmatrix}$$

5. Let U be $m \times n$ with orthonormal columns. Show that the length of $U\mathbf{x}$ is the same as the length of $\mathbf{x}$.

   SOLUTION: We show it for the squared lengths:

$$\|U\mathbf{x}\|^2 = (U\mathbf{x}) \cdot (U\mathbf{x}) = (U\mathbf{x})^T U\mathbf{x} = \mathbf{x}^T U^T U\mathbf{x} = \mathbf{x}^T I_n \mathbf{x} = \mathbf{x}^T \mathbf{x} = \|\mathbf{x}\|^2$$

6. Show that the eigenvalues of A and A^T are the same.

   SOLUTION:

   - λ is an eigenvalue of A iff $|A - \lambda I| = 0$ by definition.
   - $|A - \lambda I| = |(A - \lambda I)^T|$ by properties of the determinant.
     Simplifying further,

$$|A - \lambda I| = |A^T - \lambda I^T| = |A^T - \lambda I|$$

   - Therefore, $|A - \lambda I| = 0$ precisely when $|A^T - \lambda I| = 0$.

   Therefore, the eigenvalues of A are the same as the eigenvalues of A^T.

7. Show that if $\mathbf{u}$, $\mathbf{v}$ are eigenvectors corresponding to distinct eigenvalues, then the vectors are linearly independent.

   SOLUTION: Let A be the matrix. Then, if the vectors were linearly DEPENDENT there is a non-trivial solution to

$$c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{0}$$

   On the one hand, we can multiply both sides by A, so that

$$c_1\lambda_1\mathbf{u} + c_2\lambda_2\mathbf{v} = \mathbf{0}$$

   On the other hand, we can multiply by the non-zero eigenvalue λ_1 (if $\lambda_1 = 0$, multiply by λ_2- they both cannot be zero):

$$c_1\lambda_1\mathbf{u} + c_2\lambda_1\mathbf{v} = 0$$

   Subtracting the two equations, we have:

$$c_2(\lambda_2 - \lambda_1)\mathbf{v} = 0$$

   We see that $\lambda_2 - \lambda_1$ is not zero, so $c_2 = 0$. But, if $c_2 = 0$, then $c_1 = 0$ as well, since eigenvectors cannot be the zero vector. However, we assumed that we had a nontrivial solution to the first equation above.... This gives a contradiction. Therefore, the eigenvectors must be linearly independent.

8. Show that the coordinate mapping (from $n-$dimensional vector space V to $\mathbb{R}^n$) is onto.

   SOLUTION: Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a basis for V. Then for any vector $\mathbf{c}$ in $\mathbb{R}^n$, the associated vector in V is given by:

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$$

   which is in V because V is a vector space. Therefore, the coordinate mapping is *onto*.

9. Find the eigenvalues and eigenspaces for $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$.

SOLUTION: The characteristic equation is

$$\lambda^2 - 6\lambda + 10 = 0 \quad \Rightarrow \quad \lambda = 3 \pm i$$

If $\lambda = 3 - i$, then find the null space of $A - (3-i)I$:

$$A - (3-i)I = \begin{bmatrix} 2+i & -5 \\ 1 & -2+i \end{bmatrix}$$

Notice that $2+i$ times the second row will give you the first row. Therefore, the null space is the set of x_1, x_2 so that

$$(2+i)x_1 - 5x_2 = 0 \quad \Rightarrow \quad E_{3-i} = \text{Span}\left\{ \begin{bmatrix} 5 \\ 2+i \end{bmatrix} \right\}$$

(where I've taken x_1 to be the free variable). The other eigenspace is the complex conjugate.

10. In 5.5, we're told that if A is 2×2 with $\lambda = a - bi$, then we can write $A = PCP^{-1}$ for a certain P, C. Use this decomposition with $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$ (that is the same matrix as the previous problem).

(See p. 340, Theorem 9)