Quiz 3 Solutions

The purpose of the writing and "proof" part of the course is to get you into the text to be able to provide specific justifications to your answers (using the vocabulary of the course). In fact, one might think of these as a "gentle" introduction to proof technique- be very explicit in your thinking and write ups. (Reminder: There is someone available on Sundays and Tuesdays to help you with write ups).

1.4, 32 Could a set of 3 vectors in $\mathbb{R}^4$ span all of $\mathbb{R}^4$? Explain. What about n vectors in $\mathbb{R}^m$?

Relevant ideas: Use Theorem 4 to tie it together.

Example Solution: Let a matrix A be formed from the columns given, so that A is 4×3. There can be at most 3 pivots (one for each column), therefore there is not a pivot in every row of A. By Theorem 4, all of the statements are either true or false, and so in this case they are all false. Therefore, the columns of A do not span $\mathbb{R}^4$. Therefore, the set of original vectors cannot span $\mathbb{R}^4$.

Similarly, given a set of n vectors in $\mathbb{R}^m$, with $n < m$, let us construct a (tall) matrix A so that each vector is a column of A (therefore, A will be $m \times n$). There can be at most n pivots (one for each column), so there will not be a pivot position in every row of A. By Theorem 4, this implies that the columns of A do not span $\mathbb{R}^m$. Since the columns forms our original set, our set of vectors do not span $\mathbb{R}^m$.

1.4, 34 Suppose that A is 3×3 and $\mathbf{b}$ is a vector in $\mathbb{R}^3$ with the property that $A\mathbf{x} = \mathbf{b}$ has a unique solution. Explain why the columns of A span $\mathbb{R}^3$.

Example Solution: Since $A\mathbf{x} = \mathbf{b}$ has a unique solution for some $\mathbf{b}$, there can be no free variables. Therefore, every column of A is a pivot column, and so we can conclude that A must have 3 pivot positions. Since A has 3 rows and 3 pivot positions, there must be a pivot position in every row. Therefore, by Theorem 4, the columns of A span $\mathbb{R}^3$.

1.4 24 For the true and false, an adequate justification for "true" is to point to something in the text (a statement or theorem). If false, provide a counterexample.

(a) If $\mathbf{x}$ is a non-trivial solution to $A\mathbf{x} = \mathbf{0}$, then every entry in $\mathbf{x}$ is not zero.

FALSE. By definition, a non-trivial solution is any non-zero vector that solves the homogeneous equation. By "non-zero", we mean any vector that has *some* non-zero entry.

(b) The equation $\mathbf{x} = x_2\mathbf{u} + x_3\mathbf{v}$ with x_2 and x_3 free, (and the vectors are not multiples of each other), describes a plane through the origin.

TRUE. See the discussion, bottom of page 51 to the middle of pg. 52.

(c) The equation $A\mathbf{x} = \mathbf{b}$ is homogeneous if the zero vector is a solution.

TRUE. If the zero vector is a solution, then $\mathbf{b}$ must be the zero vector, and then the equation is the homogeneous equation.

(d) The effect of adding a vector $\mathbf{p}$ to a vector is to move the vector a direction parallel to $\mathbf{p}$

TRUE. See the discussion following Example 3.

(e) The solution set to $A\mathbf{x} = \mathbf{b}$ is obtained by translating the solution set to $A\mathbf{x} = \mathbf{0}$.

FALSE. The statement would be true if we added the statement "Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent.", as in the statement of Theorem 6.

1.5, 26 Suppose $A\mathbf{x} = \mathbf{b}$ has a solution. Explain why the solution is unique precisely when $A\mathbf{x} = \mathbf{0}$ has the trivial solution.

Main Idea: There are several ways of arguing this- For example, you might use the geometry and Theorem 6, or you may argue it more algebraically as below. Whatever you do, we're looking for the direct application of the ideas and vocabulary of the text- For example,

If $A\mathbf{x} = \mathbf{b}$ has a solution, it is unique if and only if every column of A is a pivot column. If every column of A is a pivot column, there are no free variables, and therefore the homogeneous equation has only the trivial solution (see the "fact" in the middle of pg. 50).

1.7, 40 Suppose an $m \times n$ matrix A has n pivot columns. Explain why for each $\mathbf{b} \in \mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution.

(Very similar to the last problem) If A is $m \times n$ with n pivot columns, then there are no free variables. If there are no free variables, we cannot have an infinite number of solutions.