M300
Spring 2010

Quiz 11 Solutions

- Matlab questions (from 6.6, 8 and 11)

6.6, 8 Your Matlab code should produce the cubic equation with coefficients (in increasing order):

$$0.5132 \qquad -0.0335 \qquad 0.0010$$

6.6, 11 The data should produce an eccentricity and constant:

$$\beta = 1.4509 \qquad e = 0.8111$$

Using these values and $\theta = 4.6$, then our model suggests $r = 1.33$.

- Section 6.6, 14: Show that $(\bar{x}, \bar{y})$ satisfies the least squares line:

Let β_0, β_1 be the solutions to the vector equation

$$\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

Written out, we have β_0, β_1 for which the following system of equations is satisfied:

$$\begin{array}{ll} y_1 & = \beta_0 + \beta_1 x_1 + \epsilon_1 \\ y_2 & = \beta_0 + \beta_1 x_2 + \epsilon_2 \\ \vdots & \vdots \\ y_n & = \beta_0 + \beta_1 x_n + \epsilon_n \end{array} \quad \Rightarrow \quad \sum_{i=1}^{n} y_i = n\beta_0 + \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \epsilon_i$$

Divide all by n, and note that the vector $\boldsymbol{\epsilon}$ is orthogonal to a vector of ones to conclude that its sum is zero:

$$\bar{y} = \beta_0 + \beta_1 \bar{x} + 0$$

- Section 7.1, 36:

Let B be an $n \times n$ symmetric matrix such that $B^2 = B$. Given any $\mathbf{y} \in \mathbb{R}^n$, define $\hat{\mathbf{y}} = B\mathbf{y}$ and $\mathbf{z} = \mathbf{y} = \hat{\mathbf{y}}$

1. Show that $\mathbf{z}$ is orthogonal to $\hat{\mathbf{y}}$:
   SOLUTION: We take the dot product and use the properties of B:

   $$\mathbf{z}^T\hat{\mathbf{y}} = (\mathbf{y}^T - \hat{\mathbf{y}}^T)\hat{\mathbf{y}} = (\mathbf{y}^T - (B\mathbf{y})^T)B\mathbf{y} = \mathbf{y}^T B\mathbf{y} - (B\mathbf{y})^T B\mathbf{y}$$

   Continuing, since $B^T = B$ and $B^2 = B$,

   $$\mathbf{y}^T B\mathbf{y} - \mathbf{y}^T B^T B\mathbf{y} = \mathbf{y}^T B\mathbf{y} - \mathbf{y}^T B^2\mathbf{y} = \mathbf{y}^T B\mathbf{y} - \mathbf{y}^T B\mathbf{y} = 0$$

2. Let W be the column space of B. Show that $\mathbf{y}$ is the sum of a vector in W and a vector in $W^\perp$.
   SOLUTION: From part (a), we know that using $\hat{\mathbf{y}}, \mathbf{z}$ defined in that section,

   $$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

   where $\hat{\mathbf{y}}$ is orthogonal to $\mathbf{z}$. We need only show that $\hat{\mathbf{y}} \in W$, and that will complete the argument, since then $\mathbf{z} \in W^\perp$ by part (a).
   Since $\hat{\mathbf{y}} = B\mathbf{y}$, then $\hat{\mathbf{y}}$ is a linear combination of the columns of B, and thus is in the column space of B.
   By the Orthogonal Decomposition Theorem, $\hat{\mathbf{y}}$ must be an orthogonal projection of $\mathbf{y}$ into W.