

Linear Algebra- Final Exam Review Questions

For the final exam, you may use a calculator- You may not use a calculator that performs symbolic computations (like the derivative/integral). Your calculator may be a graphing calculator (no graphing required, however).

1. Show that $\text{Row}(A) \perp \text{Null}(A)$.
2. Let A be invertible. Show that, if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent vectors, so are $A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3$. NOTE: It should be clear from your answer that you know the definition.
3. Find the line of best fit for the data:

$$\begin{array}{c|cccc} x & 0 & 1 & 2 & 3 \\ \hline y & 1 & 1 & 2 & 2 \end{array}$$

4. Let $\mathbf{v}_1, \dots, \mathbf{v}_p$ be orthonormal. If

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p$$

then show that $\|\mathbf{x}\|^2 = |c_1|^2 + \dots + |c_p|^2$. (Hint: Write the norm squared as the dot product).

5. Short answer:

- (a) If $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$, then \mathbf{u}, \mathbf{v} are orthogonal.
- (b) Let H be the subset of vectors in \mathbb{R}^3 consisting of those vectors whose first element is the sum of the second and third elements. Is H a subspace?
- (c) Explain why the image of a linear transformation $T : V \rightarrow W$ is a subspace of W
- (d) Is the following matrix diagonalizable? Explain. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 8 \\ 0 & 0 & 13 \end{bmatrix}$
- (e) If the column space of an 8×4 matrix A is 3 dimensional, give the dimensions of the other three fundamental subspaces. Given these numbers, is it possible that the mapping $\mathbf{x} \rightarrow A\mathbf{x}$ is one to one? onto?
- (f) Let Q be a matrix with 3 non-zero orthonormal columns.
 - i. Can Q have fewer than 2 rows? Why or why not?
 - ii. True or False: If Q is $m \times n$ with $m > n$, then $QQ^T = I$.

6. Find a basis for the null space, row space and column space of A , if $A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 5 & 5 \\ 0 & 0 & 3 & 3 \end{bmatrix}$

7. Find an orthonormal basis for $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ using Gram-Schmidt (you might wait until the very end to normalize all vectors at once):

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

8. Let \mathbb{P}_n be the vector space of polynomials of degree n or less. Let W_1 be the subset of \mathbb{P}_n consisting of $\mathbf{p}(t)$ so that $\mathbf{p}(0)\mathbf{p}(1) = 0$. Let W_2 be the subset of \mathbb{P}_n consisting of $\mathbf{p}(t)$ so that $\mathbf{p}(2) = 0$. Which if the two is a subspace of \mathbb{P}_n ?
9. For each of the following matrices, find the characteristic equation, the eigenvalues and a basis for each eigenspace:

$$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

10. Define $T : P_2 \rightarrow \mathbb{R}^3$ by: $T(p) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$
- Find the image under T of $p(t) = 5 + 3t$.
 - Show that T is a linear transformation.
 - Find the kernel of T . Does your answer imply that T is 1-1? Onto? (Review the meaning of these words: kernel, one-to-one, onto)
 - Find the matrix for T relative to the basis $\{1, t, t^2\}$ for P_2 . (This means that the matrix will act on the *coordinates* of p).
11. Let \mathbf{v} be a vector in \mathbb{R}^n so that $\|\mathbf{v}\| = 1$, and let $Q = I - 2\mathbf{v}\mathbf{v}^T$. Show (by direct computation) that $Q^2 = I$.
12. Let A be $m \times n$ and suppose there is a matrix C so that $AC = I_m$. Show that the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every \mathbf{b} . Hint: Consider $AC\mathbf{b}$.
13. If B has linearly dependent columns, show that AB has linearly dependent columns. Hint: Consider the null space.
14. If λ is an eigenvalue of A , then show that it is an eigenvalue of A^T .
15. Let $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, Let S be the parallelogram with vertices at $\mathbf{0}, \mathbf{u}, \mathbf{v}$, and $\mathbf{u} + \mathbf{v}$. Compute the area of S .

16. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $B = \begin{bmatrix} a+2g & b+2h & c+2i \\ d+3g & e+3h & f+3i \\ g & h & i \end{bmatrix}$, and $C = \begin{bmatrix} g & h & i \\ 2d & 2e & 2f \\ a & b & c \end{bmatrix}$.

If $\det(A) = 5$, find $\det(B)$, $\det(C)$, $\det(BC)$.

17. Let $1, t$ be two vectors in $C[-1, 1]$. Find the length between the two vectors and the cosine of the angle between them using the standard inner product (the integral). Find the orthogonal projection of t^2 onto the set spanned by $\{1, t\}$.

18. Define an *isomorphism*:

19. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -8 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \end{bmatrix} \right\}$$

Find at least two \mathcal{B} -coordinate vectors for $\mathbf{x} = [1, 1]^T$.

20. Let U, V be orthogonal matrices. Show that UV is an orthogonal matrix.

21. Find the volume of the parallelepiped formed by $\mathbf{0}$, \mathbf{a}, \mathbf{b} , \mathbf{c} , $\mathbf{a} + \mathbf{b}$, $\mathbf{c} + \mathbf{b}$, $\mathbf{c} + \mathbf{a}$, and the sum of all three.

$$\mathbf{a} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

22. Let T be a one-to-one linear transformation for a vector space V into \mathbb{R}^n . Show that for \mathbf{u}, \mathbf{v} in V , the formula:

$$\langle u, v \rangle = T(\mathbf{u}) \cdot T(\mathbf{v})$$

defines an inner product on V .

23. Describe all least squares solutions to
$$\begin{array}{rcl} x + y & = & 2 \\ x + y & = & 4 \end{array}$$

24. Let $\mathbf{u} = [5, -6, 7]^T$. Let W be the set of all vectors orthogonal to \mathbf{u} . (i) Geometrically, what is W ? (ii) Find the projection of $\mathbf{x} = [1, 2, 3]^T$ onto W . (iii) Find the distance from the vector $\mathbf{x} = [1, 2, 3]^T$ to the subspace W .