

Linear Algebra- Final Exam Review

1. Show that $\text{Row}(A) \perp \text{Null}(A)$.

SOLUTION: We can write matrix-vector multiplication in terms of the *rows* of the matrix A . If A is $m \times n$, then:

$$A\mathbf{x} = \begin{bmatrix} \text{Row}_1(A) \\ \text{Row}_2(A) \\ \vdots \\ \text{Row}_m(A) \end{bmatrix} \mathbf{x} = \begin{bmatrix} \text{Row}_1(A)\mathbf{x} \\ \text{Row}_2(A)\mathbf{x} \\ \vdots \\ \text{Row}_m(A)\mathbf{x} \end{bmatrix}$$

Each of these products is the “dot product” of a row of A with the vector \mathbf{x} .

To show the desired result, let $\mathbf{x} \in \text{Null}(A)$. Then each of the products shown in the equation above must be zero, since $A\mathbf{x} = \mathbf{0}$, so that \mathbf{x} is orthogonal to each row of A . Since the rows form a spanning set for the row space, \mathbf{x} is orthogonal to every vector in the row space.

2. Let A be invertible. Show that, if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent vectors, so are $A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3$. NOTE: It should be clear from your answer that you know the definition.

SOLUTION: We need to show that the only solution to:

$$c_1 A\mathbf{v}_1 + c_2 A\mathbf{v}_2 + c_3 A\mathbf{v}_3 = \mathbf{0}$$

is the trivial solution. Factoring out the matrix A ,

$$A(c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3) = \mathbf{0}$$

Think of the form $A\hat{\mathbf{x}} = \mathbf{0}$. Since A is invertible, the only solution to this is $\hat{\mathbf{x}} = \mathbf{0}$, which implies that the only solution to the equation above is the solution to

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 = \mathbf{0}$$

Which is (only) the trivial solution, since the vectors are linearly independent. (NOTE: Notice that if the original vectors had been linearly dependent, this last equation would have non-trivial solutions).

3. Find the line of best fit for the data:

$$\begin{array}{c|cccc} x & 0 & 1 & 2 & 3 \\ \hline y & 1 & 1 & 2 & 2 \end{array}$$

Let A be the matrix formed by a column from \mathbf{x} column of ones, then we form the normal equations $A^T A \boldsymbol{\beta} = A^T \mathbf{y}$ and solve:

$$A^T A \mathbf{x} = A^T \mathbf{y} \quad \begin{bmatrix} 14 & 6 \\ 6 & 4 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 11 \\ 6 \end{bmatrix}$$

The solution is $\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{y} = \frac{1}{10} [4, 9]^T$, so the slope is $2/5$ and the intercept is $9/10$.

4. Let $\mathbf{v}_1, \dots, \mathbf{v}_p$ be orthonormal. If

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p$$

then show that $\|\mathbf{x}\|^2 = |c_1|^2 + \dots + |c_p|^2$. (Hint: Write the norm squared as the dot product).

SOLUTION: Compute $\mathbf{x} \cdot \mathbf{x}$, and use the property that the vectors \mathbf{v}_i are orthonormal:

$$(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p) \cdot (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p) =$$

Since all dot products of the form $c_i c_k \mathbf{v}_i \cdot \mathbf{v}_k = 0$ for $i \neq k$, then the dot product simplifies to:

$$c_1^2 \mathbf{v}_1 \cdot \mathbf{v}_1 + c_2^2 \mathbf{v}_2 \cdot \mathbf{v}_2 + \dots + c_p^2 \mathbf{v}_p \cdot \mathbf{v}_p$$

And since the vectors are normalized, this gives the result:

$$\|\mathbf{x}\|^2 = c_1^2 + \dots + c_p^2$$

(Don't need the magnitudes here, since we're working with real numbers).

5. Short answer:

- (a) If $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$, then \mathbf{u}, \mathbf{v} are orthogonal.

SOLUTION: True- This is the Pythagorean Theorem.

- (b) Let H be the subset of vectors in \mathbb{R}^3 consisting of those vectors whose first element is the sum of the second and third elements. Is H a subspace?

SOLUTION: One way of showing that a subset is a subspace is to show that the subspace can be represented by the span of some set of vectors. In this case,

$$\begin{bmatrix} a+b \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Because H is the span of the given vectors, it is a subspace.

- (c) Explain why the image of a linear transformation $T : V \rightarrow W$ is a subspace of W

SOLUTION: Maybe "Prove" would have been better than "Explain", since we want to go through the three parts:

- i. $0 \in T(V)$ since $0 \in V$ and $T(0) = 0$.
- ii. Let u, v be in $T(V)$. Then there is an x, y in V so that $T(x) = u$ and $T(y) = v$. Since V is a subspace, $x+y \in V$, and therefore $T(x+y) = T(x) + T(y) = u+v$ so that $u+v \in T(V)$.
- iii. Let $u \in T(V)$. Show that $cu \in T(V)$ for all scalars c . If $u \in T(V)$, there is an x in V so that $T(x) = u$. Since V is a subspace, $cx \in V$, and $T(cx) \in T(V)$. By linearity, this means $cT(x) \in T(V)$.

(OK, that probably should not have been in the short answer section)

- (d) Is the following matrix diagonalizable? Explain. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 8 \\ 0 & 0 & 13 \end{bmatrix}$

SOLUTION: Yes. The eigenvalues are all distinct, so the corresponding eigenvectors are linearly independent.

- (e) If the column space of an 8×4 matrix A is 3 dimensional, give the dimensions of the other three fundamental subspaces. Given these numbers, is it possible that the mapping $\mathbf{x} \rightarrow A\mathbf{x}$ is one to one? onto?

SOLUTION: If the column space is 3-d, so is the row space. Therefore the null space (as a subspace of \mathbb{R}^4) is 1 dimensional and the null space of A^T is 5 dimensional.

Since the null space has more than the zero vector, $A\mathbf{x} = \mathbf{0}$ has non-trivial solutions, so the matrix mapping will not be 1-1. Since the column space is a three dimensional subspace of \mathbb{R}^8 , the mapping cannot be onto.

- (f) Let Q be a matrix with 3 non-zero orthonormal columns.

- i. Can Q have fewer than 2 rows? Why or why not?

SOLUTION: No. Since the columns are orthogonal, they are linearly independent, so the matrix must have 3 pivot columns, and so we must have (at least) 3 pivot rows.

- ii. True or False: If Q is $m \times n$ with $m > n$, then $QQ^T = I$.

SOLUTION: False. While $Q^TQ = I$, QQ^T is a projection to the column space of Q in the sense that

$$QQ^T\mathbf{x} = \text{Proj}_{\text{Col}(Q)}(\mathbf{x})$$

6. Find a basis for the null space, row space and column space of A , if $A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 5 & 5 \\ 0 & 0 & 3 & 3 \end{bmatrix}$

The basis for the column space is the set containing the first and third columns of A . A basis for the row space is the set of vectors $[1, 1, 0, 0]^T, [0, 0, 1, 1]^T$. A basis for the null space of A is $[-1, 1, 0, 0]^T, [0, 0, -1, 1]^T$.

7. Find an orthonormal basis for $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ using Gram-Schmidt (you might wait until the very end to normalize all vectors at once):

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

SOLUTION: Using Gram Schmidt (before normalization, which is OK if doing by hand), we get

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

8. Let \mathbb{P}_n be the vector space of polynomials of degree n or less. Let W_1 be the subset of \mathbb{P}_n consisting of $\mathbf{p}(t)$ so that $\mathbf{p}(0)\mathbf{p}(1) = 0$. Let W_2 be the subset of \mathbb{P}_n consisting of $\mathbf{p}(t)$ so that $\mathbf{p}(2) = 0$. Which of the two is a subspace of \mathbb{P}_n ?

SOLUTION: We check the properties-

- Is 0 in the subspace?

For both W_1 and W_2 , the zero polynomial satisfies both.

- Is the subspace closed under addition?

For W_1 , let $p(t)$ be a polynomial such that $p(0)p(1) = 0$, and let $h(t)$ be another polynomial with that property, $h(0)h(1) = 0$.

Does that imply that $g(t) = p(t) + h(t)$ has the desired property?

$$g(0)g(1) = (p(0)+h(0))(p(1)+h(1)) = p(0)p(1)+p(0)h(1)+h(0)p(1)+h(0)h(1) = p(0)h(1)+h(0)p(1)$$

For example, if $h(0) = 1$, $h(1) = 0$, $p(0) = 0$, $p(1) = 1$, then this quantity is not zero. Therefore, W_1 is not closed under addition.

However, you can check that W_2 is closed under addition: Let $p(t), h(t)$ be two functions in W_2 . Then $g(t) = p(t) + h(t)$ satisfies the property that

$$g(2) = p(2) + h(2) = 0 + 0 = 0$$

- Similarly, W_2 is closed under scalar multiplication- If $g(t) = cp(t)$, then $g(2) = cp(2) = c \cdot 0 = 0$.

Therefore, W_2 is a subspace, and W_1 is not.

9. For each of the following matrices, find the characteristic equation, the eigenvalues and a basis for each eigenspace:

$$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

SOLUTION: For matrix A , $\lambda = 3, 5$. Eigenvectors are $[1, 1]^T$ and $[2, 1]^T$, respectively.

For matrix B , for $\lambda = 3+i$, an eigenvector is $[1, i]^T$. The other eigenvalue and eigenvector are the complex conjugates.

For matrix C , expand along the 2d row. $\lambda = 2$ is a double eigenvalue with eigenvectors $[0, 1, 0]^T$ and $[1, 0, 1]^T$. The third eigenvalue is $\lambda = 0$ with eigenvector $[-1, 0, 1]^T$.

10. Define $T : P_2 \rightarrow \mathbb{R}^3$ by: $T(p) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$

- (a) Find the image under T of $p(t) = 5 + 3t$.

SOLUTION: $[2, 5, 8]^T$

- (b) Show that T is a linear transformation.

SOLUTION: We show it using the definition.

- i. Show that $T(p + q) = T(p) + T(q)$:

$$T(p + q) = \begin{bmatrix} p(-1) + q(-1) \\ p(0) + q(0) \\ p(1) + q(1) \end{bmatrix} = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix} + \begin{bmatrix} q(-1) \\ q(0) \\ q(1) \end{bmatrix} = T(p) + T(q)$$

- ii. Show that $T(cp) = cT(p)$ for all scalars c .

$$T(cp) = \begin{bmatrix} cp(-1) \\ cp(0) \\ cp(1) \end{bmatrix} = c \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix} = cT(p)$$

- (c) Find the kernel of T . Does your answer imply that T is 1 – 1? Onto? (Review the meaning of these words: kernel, one-to-one, onto)

SOLUTION:

Since the kernel is the set of elements in the domain that map to zero, let's see what the action of T is on an arbitrary polynomial. An arbitrary vector in P_2 is: $p(t) = at^2 + bt + c$, and:

$$T(at^2 + bt + c) = \begin{bmatrix} a - b + c \\ c \\ a + b + c \end{bmatrix}$$

For this to be the zero vector, $c = 0$. Then $a - b = 0$ and $a + b = 0$, so $a = 0, b = 0$. Therefore, the only vector mapped to zero is the zero vector.

Side Remark: Recall that for any linear function T , if we are solving $T(x) = y$, then the solution can be written as $x = x_p + x_h$, where x_p is the particular solution (it solves $T(x_p) = y$), and $T(x_h) = 0$ (we said x_h is the homogeneous part of the solution). So the equation $T(x) = y$ has at most one solution iff the kernel is only the zero vector (if T was realized as a matrix, we get our familiar setting).

Therefore, T is 1 – 1. The mapping T will also be onto (see the next part).

- (d) Find the matrix for T relative to the basis $\{1, t, t^2\}$ for P_2 . (This means that the matrix will act on the *coordinates* of p).

You can verify that the matrix representation for T (using the basis in the order $at^2 + bt + c$) is:

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

since multiplying this matrix by $[a, b, c]^T$ will give the previous result.

From this we see that the matrix is invertible, so the mapping is both 1-1 and onto (as a mapping from \mathbb{R}^3 to \mathbb{R}^3).

11. Let \mathbf{v} be a vector in \mathbb{R}^n so that $\|\mathbf{v}\| = 1$, and let $Q = I - 2\mathbf{v}\mathbf{v}^T$. Show (by direct computation) that $Q^2 = I$.

SOLUTION: This problem is to practice matrix algebra:

$$Q^2 = (I - 2\mathbf{v}\mathbf{v}^T)(I - 2\mathbf{v}\mathbf{v}^T) = I^2 - 2I\mathbf{v}\mathbf{v}^T - 2\mathbf{v}\mathbf{v}^T I + 4\mathbf{v}\mathbf{v}^T \mathbf{v}\mathbf{v}^T = I - 4\mathbf{v}\mathbf{v}^T + 4\mathbf{v}(1)\mathbf{v}^T = I$$

12. Let A be $m \times n$ and suppose there is a matrix C so that $AC = I_m$. Show that the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every \mathbf{b} . Hint: Consider $AC\mathbf{b}$.

SOLUTION: Using the hint, we see that $AC\mathbf{b} = \mathbf{b}$. Therefore, given an arbitrary vector \mathbf{b} , the solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = C\mathbf{b}$.

13. If B has linearly dependent columns, show that AB has linearly dependent columns. Hint: Consider the null space.

SOLUTION: If B has linearly dependent columns, then the equation $B\mathbf{x} = \mathbf{0}$ has non-trivial solutions. Therefore, the equation $AB\mathbf{x} = \mathbf{0}$ has (the same) non-trivial solutions, and the columns of AB must be linearly dependent.

14. If λ is an eigenvalue of A , then show that it is an eigenvalue of A^T .

SOLUTION: Use the properties of determinants. Given

$$|A - \lambda I| = |(A - \lambda I)^T| = |A^T - \lambda I^T| = |A^T - \lambda I|$$

the solutions to $|A - \lambda I| = 0$ and $|A^T - \lambda I| = 0$ are exactly the same.

15. Let $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, Let S be the parallelogram with vertices at $\mathbf{0}$, \mathbf{u} , \mathbf{v} , and $\mathbf{u} + \mathbf{v}$. Compute the area of S .

SOLUTION: The area of the parallelogram formed by two vectors in \mathbb{R}^2 is the determinant of the matrix whose columns are those vectors. In this case, that would be 4.

16. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $B = \begin{bmatrix} a + 2g & b + 2h & c + 2i \\ d + 3g & e + 3h & f + 3i \\ g & h & i \end{bmatrix}$, and $C = \begin{bmatrix} g & h & i \\ 2d & 2e & 2f \\ a & b & c \end{bmatrix}$.

If $\det(A) = 5$, find $\det(B)$, $\det(C)$, $\det(BC)$.

SOLUTION: This question reviews the relationship between the determinant and row operations. The determinant of B is 5. The determinant of C is -10 . The determinant of BC is -50 .

17. Let $1, t$ be two vectors in $C[-1, 1]$. Find the length between the two vectors and the cosine of the angle between them using the standard inner product (the integral). Find the orthogonal projection of t^2 onto the set spanned by $\{1, t\}$.

SOLUTION:

- The length between the vectors is:

$$\sqrt{\langle (1-t), (1-t) \rangle} = \sqrt{\int_{-1}^1 (1-t)^2 dt} = \sqrt{\left. \frac{-1}{3}(1-t)^3 \right|_{-1}^1} = \sqrt{\frac{8}{3}}$$

18. Define an *isomorphism*: A one-to-one linear transformation between vector spaces (see p. 251)

NOTE: An isomorphism was the critical piece to understanding when two vector spaces had the same “form”- For example, a plane through the origin in \mathbb{R}^3 and the plane \mathbb{R}^2 are not equal, but they are isomorphic; the isomorphism takes a point of the plane and returns its coordinates- That is, the plane in \mathbb{R}^3 is the span of two vectors in \mathbb{R}^3 , so every point on the plane is a linear combination of those two. The point in \mathbb{R}^2 that we refer to is the ordered pair of weights from the linear combination.

As another example, if the plane is spanned by \mathbf{u} and \mathbf{v} in vector space V , and \mathbf{x} is on the plane so that $\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v}$, then the isomorphism takes $\mathbf{x} \in V$ and gives $(c_1, c_2) \in \mathbb{R}^2$.

19. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -8 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \end{bmatrix} \right\}$$

Find at least two \mathcal{B} -coordinate vectors for $\mathbf{x} = [1, 1]^T$.

SOLUTION: Row reduce to find \mathbf{x} as a linear combination of the vectors in \mathcal{B} :

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ -3 & -8 & 7 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -5 & 5 \\ 1 & 1 & 1 & -2 \end{bmatrix}$$

If we label the weights as c_1, c_2 and c_3 , then

$$\begin{aligned} c_1 &= 5 + 5t \\ c_2 &= -2 - t \\ c_3 &= t \end{aligned}$$

Therefore, we could form \mathbf{x} using the weights $(5, -2, 0)$, or $(10, -3, 1)$ or any combination given.

NOTE: The columns do NOT form a basis for \mathbb{R}^2 , but they do form a spanning set for \mathbb{R}^2 . If the columns formed a basis, the weights for the linear combination would be unique (no free variables), but in this case, the expansion of \mathbf{x} in this basis was not unique.

20. Let U, V be orthogonal matrices. Show that UV is an orthogonal matrix.

SOLUTION: This question deals with the *definition* of an orthogonal matrix: A square matrix such that $U^T = U^{-1}$. First, if the product UV is defined, then U and V are both $n \times n$ for some n .

Secondly, since U, V are each invertible, then so is UV . Furthermore,

$$(UV)^{-1} = V^{-1}U^{-1} = V^T U^T = (UV)^T$$

Therefore, UV is an orthogonal matrix.

21. Find the volume of the parallelepiped formed by $\mathbf{0}$, \mathbf{a} , \mathbf{b} , \mathbf{c} , $\mathbf{a} + \mathbf{b}$, $\mathbf{c} + \mathbf{b}$, $\mathbf{c} + \mathbf{a}$, and the sum of all three.

$$\mathbf{a} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

SOLUTION: The volume is the absolute value of the determinant. In this case, expand along the second row, and get 3.

22. Let T be a one-to-one linear transformation for a vector space V into \mathbb{R}^n . Show that for \mathbf{u}, \mathbf{v} in V , the formula:

$$\langle u, v \rangle = T(\mathbf{u}) \cdot T(\mathbf{v})$$

defines an inner product on V .

SOLUTION: This was a homework problem from 6.7. We want to check the properties of the inner product, which are: (i) Symmetry, (ii) and (iii) Linear in the first coordinate, and (iv) Inner product of a vector with itself is non-negative (and the special case of 0).

- (a) $\langle \mathbf{u}, \mathbf{v} \rangle = T(\mathbf{u}) \cdot T(\mathbf{v}) = T(\mathbf{v}) \cdot T(\mathbf{u}) = \langle \mathbf{v}, \mathbf{u} \rangle$, where the second equality is true because the regular dot product is symmetric.

(b)

$$\langle \mathbf{u} + \mathbf{w}, \mathbf{v} \rangle = T(\mathbf{u} + \mathbf{w}) \cdot T(\mathbf{v}) = (T(\mathbf{u}) + T(\mathbf{w})) \cdot T(\mathbf{v}) = T(\mathbf{u}) \cdot T(\mathbf{v}) + T(\mathbf{w}) \cdot T(\mathbf{v})$$

(c)

$$\langle c\mathbf{u}, \mathbf{v} \rangle = T(c\mathbf{u}) \cdot T(\mathbf{v}) = cT(\mathbf{u}) \cdot T(\mathbf{v})$$

(d)

$$\langle \mathbf{u}, \mathbf{u} \rangle = T(\mathbf{u}) \cdot T(\mathbf{u}) = \|T(\mathbf{u})\|^2$$

The dot product of a vector with itself is always non-negative. Furthermore, by the same equation, if $\langle u, u \rangle = 0$, then u must be the zero vector.

23. Describe all least squares solutions to
$$\begin{array}{rcl} x + y & = & 2 \\ x + y & = & 4 \end{array}$$

SOLUTION: Interesting to think about- In the plane, these are two parallel lines (each has a slope of -1 , one has an intercept at 2, the other at 4).

Using linear algebra, we have

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \Leftrightarrow \quad A\mathbf{x} = \mathbf{b}$$

We cannot use the normal equations, because A does not have full rank (a rank of 2). However, if we project \mathbf{b} into the column space of A (which is the span of $[1, 1]^T$), then

we can solve the system (and in fact, we'll have an infinite number of solutions since there will be a free variable):

$$\text{Proj}_{\text{Col}(A)}(\mathbf{b}) = \frac{2+4}{1+1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Now solve the system $A\hat{\mathbf{x}} = [3, 3]^T$, which is:

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

If we let the free variable be $\hat{y} = t$, then $\hat{x} = 3 - t$. Notice that this set of points represents the line $\hat{y} = -\hat{x} + 3$, which is the line right down the middle between the other two lines!

24. Let $\mathbf{u} = [5, -6, 7]^T$. Let W be the set of all vectors orthogonal to \mathbf{u} . (i) Geometrically, what is W ? (ii) Find the projection of $\mathbf{x} = [1, 2, 3]^T$ onto W . (iii) Find the distance from the vector $\mathbf{x} = [1, 2, 3]^T$ to the subspace W .

SOLUTIONS:

- W is the plane in \mathbb{R}^3 going through the origin whose normal vector (in the sense of Calc 3) is \mathbf{u} .
- We can write $\mathbf{x} = \hat{\mathbf{x}} + \mathbf{z}$, where $\hat{\mathbf{x}}$ is the projection onto \mathbf{u} , then \mathbf{z} will be the desired vector in W :

$$\hat{\mathbf{x}} = \left(\frac{\mathbf{x} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u} = \frac{5 - 12 + 21}{25 + 36 + 49} \begin{bmatrix} 5 \\ -6 \\ 7 \end{bmatrix} = \begin{bmatrix} 7/11 \\ -42/55 \\ 49/55 \end{bmatrix} \approx \begin{bmatrix} 0.6364 \\ -0.7636 \\ 0.8909 \end{bmatrix} \Rightarrow \mathbf{z} \approx \begin{bmatrix} 0.36 \\ 2.76 \\ 2.11 \end{bmatrix}$$

- The distance is then $\|\mathbf{x} - \mathbf{z}\| = \|\hat{\mathbf{x}}\| \approx 1.33$