Review Questions: Sections 1.1-2.3

The first exam will cover sections 1.1-1.5, 1.7-1.9, 2.1-2.3. You should be familiar with the homework. Typical exam-style questions are given below. There are many more questions here than will be on the exam, which should take about 50 minutes.

No calculators will be allowed on the exam.

Definitions and Basic Theorems

Definitions should be memorized, and you should be able to give the result of the basic theorems below, given their hypotheses as below.

1. Finish the definition:
   (a) A **linear combination** of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is:
   (b) A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are said to be **linearly independent** if:
   (c) The **span** of a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is:
   (d) An $n \times n$ matrix A is **invertible** if:
   (e) A system of equations is **inconsistent** if:
   (f) A system of equations is **homogeneous** if:
   (g) Two matrices are **row equivalent** if:
   (h) A transformation $T : X \to Y$ is said to be **linear** if:
   (i) An **elementary matrix** E is:
2. Given the **definition** of the operation shown:
   - Matrix-vector multiplication: $A\mathbf{x} =$
   - Matrix-matrix multiplication: $AB =$
3. What information about $T : \mathbb{R}^n \to \mathbb{R}^m$ do we need to know in order to compute the standard matrix for the transformation?
4. Give the **definitions** for each (for general functions f):
   - A function $f : X \to Y$ is **1-1** if:
   - A function $f : X \to Y$ is **onto** if:
5. Fill in the blanks for the Existence and Uniqueness Theorem. Your answers should refer to pivot columns:
   - A linear system is consistent if and only if:
   - Furthermore, if the system is consistent, the solution is unique if:
6. Give three statements that are logically equivalent to saying that A has a pivot in every row. I'll give some hints so you can fill in the blanks:

- For each $\mathbf{b} \in \mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ _____
- _____ of A span $\mathbb{R}^m$
- Each $\mathbf{b}$ is a linear combination of _____ of A
- The mapping $\mathbf{x} \to A\mathbf{x}$ will be _____(choose from 1-1, onto, both or neither)

7. Similar to the last problem, give two statements that are logically equivalent to saying that: A has a pivot in every column.

8. Give the formula for the inverse of a 2×2 matrix A.

Computational Questions

1. Let $\mathbf{a}_3 = 2\mathbf{a}_1 - 3\mathbf{a}_2$. Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3]$. If A is 3×3 with 2 pivots, write the solution to $A\mathbf{x} = \mathbf{0}$ in parametric form.

2. Write the equation of a plane that spans $[1, 2, 3, 4]^T$, $[1, -1, -1, 1]^T$ and has been translated by $[3, 0, 0, 1]^T$

3. Find the general solution (in parametric vector form) to the system:

$$\begin{aligned} x_1 + 3x_2 + x_3 + x_4 &= -1 \\ -2x_1 - 6x_2 - x_3 &= 5 \\ x_1 + 3x_2 + 2x_3 + 3x_4 &= 2 \end{aligned}$$

4. Suppose the solution set of a certain system of linear equations is given by $x_1 = 5 + 4x_4$, $x_2 = -2 + 7x_4$ with $x_3 = 2 + x_4$ and x_4 is a free variable.

   (a) Use vectors to describe the solution set as a (parametric) line in $\mathbb{R}^4$.

   (b) Was the original system homogeneous? If not, give the solution to the homogeneous system of equations, if you have enough information.

5. Show that the mapping T is not linear: $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 - 2x_2 \\ 3x_2 + 1 \end{bmatrix}$

6. Find two different matrices D so that $AD = I_2$, if $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ Is the mapping $\mathbf{x} \to A\mathbf{x}$ going to be $1 - 1$? Onto $\mathbb{R}^3$?

7. Given the matrix A below, explain whether or not the system $A\mathbf{x} = \mathbf{b}$ has a solution in terms of h. If there are restrictions on $\mathbf{b}$, give them.

$$A = \begin{bmatrix} 1 & -3 \\ 2 & -h \end{bmatrix}$$

8. Let A be an 3×3 matrix. Construct the elementary matrix E_1 to perform the row operation $3r_2 + r_1 \to r_1$. Compute E_1^{-1} as well.

9. Let A be a 3×4 matrix, let $\boldsymbol{y}_1, \boldsymbol{y}_2$ be vectors, and let $\boldsymbol{w} = \boldsymbol{y}_1 + \boldsymbol{y}_2$. Suppose that $\boldsymbol{y}_1 = A\boldsymbol{x}_1$ and $\boldsymbol{y}_2 = A\boldsymbol{x}_2$ for some vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.
   (a) What size must $\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{x}_1, \boldsymbol{x}_2$ be?
   (b) Does $A\boldsymbol{x} = \boldsymbol{w}$ have a solution? Why or why not?

10. Suppose that:
$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 0 \end{bmatrix}, \quad \text{and } T\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
Find a matrix A so that $T(\boldsymbol{x}) = A\boldsymbol{x}$.

11. Suppose that for an $n \times n$ matrix A, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for each $\boldsymbol{b}$ in $\mathbb{R}^n$. Explain why A must be invertible. Does your answer change if A is $n \times m$?

12. Let A and B be given below. (a) Find the second columns of AB (without computing everything). (b) Find the $(3, 1)$ entry of $B^T A$. (c) Determine the inverse of BB^T, if possible.
$$A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}$$

13. Determine the matrix for the linear transformation T given below: $T(x_1, x_2, x_3, x_4) = 3x_1 - 4x_2 + 8x_4$

14. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$, where $T(\boldsymbol{e}_1) = (1, 4)$, $T(\boldsymbol{e}_2) = (-2, 9)$, and $T(\boldsymbol{e}_3) = (3, -8)$. Find a matrix A so that $T(\mathbf{x}) = A\mathbf{x}$.

15. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ so that $T(\mathbf{x}) = A\mathbf{x}$, where
$$A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \\ -1 & -3 \end{bmatrix}$$
Is T 1-1? Explain. Is T onto? Explain.

16. Suppose that A, B, C are $n \times n$ matrices with B invertible, and $I - BAB^{-1} = C$. Solve this equation for A. Be sure to show your work, and if you invert a matrix, explain why it is invertible.

Discussion Questions

These types of questions are more theoretical in nature. You do not have to refer to specific theorem numbers in your justifications, but you should note the existence of a theorem, if there is one to help. You should not argue "naively", or from first principles- Use the material that we have learned.

1. If $T : X \to Y$ and $S : Y \to Z$ are linear transformations, show that the composition is a linear transformation from X to Z.

2. Let A be $m \times n$. Let C be $n \times m$ so that $CA = I_n$. Show that the equation $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution. *NOTE: You cannot assume that $n = m$, because that might not be the case!*

3. Let A be $m \times n$. Let D be an $n \times m$ matrix so that $AD = I_m$. Show that $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$.

4. Show that if the columns of B are linearly dependent, then so are the columns of AB.

5. Let A, B be $n \times n$. Show that if AB is invertible, then so is A.

6. Let A be $n \times n$. If the equation $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution, do the columns of A span $\mathbb{R}^n$? Why or why not? Is your answer different if A is $n \times m$?

7. Let T be a linear transformation. Show that if $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ are linearly dependent vectors, then $\{T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), T(\boldsymbol{v}_3)\}$ are linearly dependent vectors.

8. If H is 7×7 matrix and $H\boldsymbol{x} = \boldsymbol{v}$ is consistent for every $\boldsymbol{v}$ in $\mathbb{R}^7$, then is it possible for $H\boldsymbol{x} = \boldsymbol{v}$ to have *more* than one solution for some $\boldsymbol{v} \in \mathbb{R}^7$? Why or why not?

9. Let A be invertible. Show that, if $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent, then $\{A\boldsymbol{v}_1, A\boldsymbol{v}_2, A\boldsymbol{v}_3\}$ is linearly independent. Is this true if A is not invertible?

10. If A is $n \times n$ and invertible, show that A^T is invertible. (Hint: What should the inverse of A^T be? Show that your answer works).

True or False (and explain)?

1. If A is invertible, then the elementary row operations that reduce A to the identity also reduce A^{-1} to the identity.

2. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each vector in $\mathbb{R}^n$ maps onto a unique vector in $\mathbb{R}^m$.

3. $A^T + B^T = (A + B)^T$

4. $(AB)C = (AC)B$

5. $((AB)^T)^{-1} = A^{-1}B^{-1}$ (you may assume A, B are invertible)

6. If A is 5×5, and the columns of A do not span $\mathbb{R}^5$, it is possible that A is invertible.

7. A linear transformation preserves the operations of vector addition and scalar multiplication.

8. If $A\mathbf{x} = \mathbf{b}$ has more than 1 solution, so does $A\mathbf{x} = \mathbf{0}$.

9. In some cases, it is possible for four vectors to span $\mathbb{R}^5$.

10. If A, B are row equivalent $m \times n$ matrices, and if the columns of A span $\mathbb{R}^m$, then so do the columns of B.