Review Solutions for Exam 1

Definitions and Basic Theorems

1. Finish the definition:

   (a) A **linear combination** of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is: any vector of the form

   $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

   (b) A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are said to be **linearly independent** if: the only solution to

   $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = 0$$

   is the trivial solution, $c_1 = c_2 = \cdots = c_n = 0$.

   (c) The **span** of a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is: the set of all linear combinations of the vectors.

   (d) An $n \times n$ matrix A is **invertible** if: there is a matrix C such that $AC = I$.
   NOTE: Since A is square, it suffices to have $AC = I$ or $DA = I$, we don't need both.

   (e) A system of equations is **inconsistent** if: there is no solution that satisfies all of the equations.

   (f) A system of equations is **homogeneous** if: the equations are set to zero.

   (g) Two matrices are **row equivalent** if: you can change one into the other by a sequence of row operations.

   (h) A transformation $T : X \to Y$ is said to be **linear** if: $T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2)$, for all $\mathbf{x}_1, \mathbf{x}_2$ in X, and $T(c\mathbf{x}) = cT(\mathbf{x})$ for all $\mathbf{x} \in X$ and all scalars c.

   (i) An **elementary matrix** E is: a matrix formed by using one row operation on the identity matrix.

2. Given the **definition** of the operation shown:

   - Matrix-vector multiplication: $A\mathbf{x} = x_1\mathbf{a}_1 + \cdot + x_n\mathbf{a}_n$, where $\mathbf{a}_i$ is the i^{th} column of the matrix A.
   - Matrix-matrix multiplication: $AB = [A\mathbf{b}_1 \;\; A\mathbf{b}_2 \;\; \cdots \;\; A\mathbf{b}_k]$ where $\mathbf{b}_i$ is the i^{th} column of B.

3. What information about $T : \mathbb{R}^n \to \mathbb{R}^m$ do we need to know in order to compute the standard matrix for the transformation?

   SOLUTION: The best information to know is where T sends the standard basis vectors. That is, $T(\mathrm{e}_1), \ldots, T(\mathrm{e}_n)$. That's because these are the columns of the matrix. It is possible to use any linearly independent set of n vectors, but we would have some work to do to find the matrix.

4. Give the **definitions** for each (for general functions f):

   - A function $f : X \to Y$ is **1-1** if: $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, or you could say that the preimage of every $y \in Y$ in the range is a single point in the domain.
   - A function $f : X \to Y$ is **onto** if: every $y \in Y$ came from some $xinX$. That is, $y = f(x)$ for every $y \in Y$.

5. Fill in the blanks for the Existence and Uniqueness Theorem. Your answers should refer to pivot columns:

   - A linear system is consistent if and only if: the rightmost column of the augmented matrix is not a pivot column.
   - Furthermore, if the system is consistent, the solution is unique if: every column of the coefficient matrix is a pivot column.

6. Give three statements that are logically equivalent to saying that A has a pivot in every row. I'll give some hints so you can fill in the blanks:

   - For each $\mathbf{b} \in \mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b}$.
   - The columns of A span $\mathbb{R}^m$

- Each $\mathbf{b}$ is a linear combination of the columns of A
- The mapping $\mathbf{x} \to A\mathbf{x}$ will be ONTO.

7. Similar to the last problem, give two statements that are logically equivalent to saying that: A has a pivot in every column.

   SOLUTION: Here are some options:

   (a) The mapping $\mathbf{x} \to A\mathbf{x}$ is 1-1.

   (b) For every $\mathbf{b} \in \mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has AT MOST 1 solution.

   (c) The homogeneous equation has only the trivial solution.

   (d) The columns of A are linearly independent.

8. Give the formula for the inverse of a 2×2 matrix A.

   SOLUTION: If A is given as below, the inverse follows:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Computational Questions

1. Let $\mathbf{a}_3 = 2\mathbf{a}_1 - 3\mathbf{a}_2$. Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3]$. If A is 3×3 with 2 pivots, write the solution to $A\mathbf{x} = \mathbf{0}$ in parametric form.

   SOLUTION: If we have two pivots, then there is one free variable. Therefore, we know the solution set is a line through the origin. Also, since

$$2\mathbf{a}_1 - 3\mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}$$

   then the solution set must be: $s[2, -3, -1]^T$, where s is a free value (parameter).

2. Write the equation of a plane that spans $[1, 2, 3, 4]^T$, $[1, -1, -1, 1]^T$ and has been translated by $[3, 0, 0, 1]^T$

   SOLUTION:

$$\begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

3. Find the general solution (in parametric vector form) to the system:

$$\begin{aligned} x_1 + 3x_2 + x_3 + x_4 &= -1 \\ -2x_1 - 6x_2 - x_3 &= 5 \\ x_1 + 3x_2 + 2x_3 + 3x_4 &= 2 \end{aligned}$$

   SOLUTION: Row reduction gets you the matrix to the left; solving gives you the parametric equation on the right:

$$\left[\begin{array}{rrrr|r} 1 & 3 & 1 & 1 & -1 \\ -2 & -6 & -1 & 0 & 5 \\ 1 & 3 & 2 & 3 & 2 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 3 & 0 & -1 & -4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \boldsymbol{x} = \begin{bmatrix} -4 \\ 0 \\ 3 \\ 0 \end{bmatrix} + r \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

4. Suppose the solution set of a certain system of linear equations is given by $x_1 = 5 + 4x_4$, $x_2 = -2 + 7x_4$ with $x_3 = 2 + x_4$ and x_4 is a free variable.

   (a) Use vectors to describe the solution set as a (parametric) line in $\mathbb{R}^4$.

   SOLUTION: $\mathbf{x} = [5, -2, 2, 0]^T + x_4[4, 7, 1, 1]^T$

(b) Was the original system homogeneous? If not, give the solution to the homogeneous system of equations, if you have enough information.

SOLUTION: No, the system was not homogeneous since 0 is not a solution. However, we know all solutions are of the form $\mathbf{x}_p + \mathbf{x}_h$, where the parameters are in $\mathbf{x}_h$. Therefore, we can think of $[5, -2, 2, 0]^T$ as the particular part of the solution to $A\mathbf{x} = \mathbf{b}$, and $x_4[4, 7, 1, 1]^T$ is the homogeneous part of the solution.

5. Show that the mapping T is not linear: $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 - 2x_2 \\ 3x_2 + 1 \end{bmatrix}$

Quick method: $T(0) = [0, 1]^T \neq [0, 0]^T$, therefore the mapping is not linear.

6. Find two different matrices D so that $AD = I_2$, if $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ Is the mapping $\mathbf{x} \to A\mathbf{x}$ going to be $1-1$? Onto $\mathbb{R}^3$?

SOLUTION: To find the matrix D, you might try guess-and-check, or set up a system of equations. Since D will need to be 3×2, we have:

$$AD = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_1 & d_4 \\ d_2 & d_5 \\ d_3 & d_6 \end{bmatrix} = I_2 \Rightarrow \begin{array}{rl} d_1 + d_2 - d_3 & = 1 \\ d_1 \qquad + d_3 & = 0 \end{array} \quad \text{and} \quad \begin{array}{rl} d_4 + d_5 - d_6 & = 0 \\ d_4 \qquad + d_6 & = 1 \end{array}$$

Solve these for the 6 values of D, and we have an infinite number of choices. Here are a couple of matrices:

$$D = \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 1 & 1 \end{bmatrix}$$

For the second part of the question, we note that A has two pivots, and the columns of A are in $\mathbb{R}^2$, so the columns span all of $\mathbb{R}^2$. Therefore, the mapping $\mathbf{x} \to A\mathbf{x}$ is onto. However, the mapping is not $1-1$ since we will always have a free variable.

(This agrees with the homework from 2.1, where we said that if $AD = I$, we can always find a solution to $A\mathbf{x} = \mathbf{b}$ by taking $\mathbf{x} = D\mathbf{b}$).

7. Given the matrix A below, explain whether or not the system $A\mathbf{x} = \mathbf{b}$ has a solution in terms of h. If there are restrictions on $\mathbf{b}$, give them.

$$A = \begin{bmatrix} 1 & -3 \\ 2 & -h \end{bmatrix}$$

SOLUTION: Row reduce:

$$\left[\begin{array}{cc|c} 1 & -3 & b_1 \\ 2 & -h & b_2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & b_1 \\ 0 & 6-h & b_2 - 2b_1 \end{array}\right]$$

Therefore, if $h \neq 6$, we have a unique solution for all vectors $\mathbf{b}$.

If $h = 6$, the system is consistent only when $b_2 = 2b_1$, and in that case we have an infinite number of solutions to the system. Otherwise, if $b_2 \neq 2b_1$, the system is inconsistent.

8. Let A be an 3×3 matrix. Construct the elementary matrix E_1 to perform the row operation $3r_2 + r_1 \to r_1$. Compute E_1^{-1} as well.

$$E_1 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_1^{-1} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

9. Let A be a 3×4 matrix, let $\boldsymbol{y}_1, \boldsymbol{y}_2$ be vectors, and let $\boldsymbol{w} = \boldsymbol{y}_1 + \boldsymbol{y}_2$. Suppose that $\boldsymbol{y}_1 = A\boldsymbol{x}_1$ and $\boldsymbol{y}_2 = A\boldsymbol{x}_2$ for some vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.

(a) What size must $\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{x}_1, \boldsymbol{x}_2$ be?

SOLUTION: The vectors $\mathbf{x}_1, \mathbf{x}_2$ are in $\mathbb{R}^4$ and $\mathbf{y}_1, \mathbf{y}_2$ are in $\mathbb{R}^3$

(b) Does $A\boldsymbol{x} = \boldsymbol{w}$ have a solution? Why or why not?

SOLUTION: Yes-

$$\mathbf{w} = \mathbf{y}_1 + \mathbf{y}_2 = A\mathbf{x}_1 + A\mathbf{x}_2 = A(\mathbf{x}_1 + \mathbf{x}_2)$$

so the solution to $A\mathbf{x} = \mathbf{w}$ is $\mathbf{x} + \mathbf{x}_1 + \mathbf{x}_2$.

10. Suppose that:

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 0 \end{bmatrix}, \text{ and } T\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Find a matrix A so that $T(\boldsymbol{x}) = A\boldsymbol{x}$.

SOLUTION: In this case, we don't know $T(\mathbf{e}_1$, $T(\mathbf{e}_2)$, so we'll try something different. Let the matrix A be given below, so that we need to find a, b, c, d:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

These lead us to the system of equations:

$$\begin{array}{rl} a + 2b & = -3 \\ -2a + b & = 1 \end{array} \qquad \begin{array}{rl} c + 2d & = 0 \\ -2c + d & = 2 \end{array} \Rightarrow \quad A = \begin{bmatrix} -1 & -1 \\ -4/5 & 2/5 \end{bmatrix}$$

11. Suppose that for an $n \times n$ matrix A, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for each $\boldsymbol{b}$ in $\mathbb{R}^n$. Explain why A must be invertible. Does your answer change if A is $n \times m$?

This question should go in the next section. In any event, there are multiple ways of answering this question. Here is one way to do it:

If the equation has a solution for every $\mathbf{b}$, then there is a pivot in every row of A. Since A is square, A has exactly n pivots, so A is invertible by the invertible matrix theorem.

If A is not square, there may be extra columns in A that do not correspond to pivot columns (A must be square or wide, but not tall since there is a pivot in every row). Thus, the mapping may not be 1-1, in which case the matrix is not invertible.

12. Let A and B be given below. (a) Find the second column of AB (without computing everything). (b) Find the $(3, 1)$ entry of B^TA. (c) Determine the inverse of BB^T, if possible.

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}$$

SOLUTION: (a) the second column of AB is $A\mathbf{b}_2 = [20, -49]^T$.

(b) The (3,1) entry of B^TA uses the third row of B^T (which is the third column of B "dotted" with the 1st column of A: $[-1, 3][1, -2]^T = -7$

(c) The inverse of BB^T is given by (use the formula for 2×2 matrix):

$$(BB^T)^{-1} = \frac{1}{27}\begin{bmatrix} 126 & 27 \\ 27 & 6 \end{bmatrix}$$

13. Determine the matrix for the linear transformation T given below: $T(x_1, x_2, x_3, x_4) = 3x_1 - 4x_2 + 8x_4$

SOLUTION: You could think about $T(\mathbf{e}_i)$, or just reason it out. Note that $T : \mathbb{R}^4 \to \mathbb{R}$, so A is 1×4, and

$$A = [3, -4, 0, 8]$$

14. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$, where $T(\boldsymbol{e}_1) = (1, 4)$, $T(\boldsymbol{e}_2) = (-2, 9)$, and $T(\boldsymbol{e}_3) = (3, -8)$. Find a matrix A so that $T(\mathbf{x}) = A\mathbf{x}$.

SOLUTION: First, note the size of A should be 2×3, and we already have the action of T on the standard basis vectors. The matrix is $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 9 & -8 \end{bmatrix}$

15. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ so that $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \\ -1 & -3 \end{bmatrix}$$

Is T 1-1? Explain. Is T onto? Explain.

SOLUTION: Examining the matrix, we see that there are two pivots. Therefore, if the system is consistent (and the last row of the rref form of A will be all zeros), there is exactly one solution.

This corresponds to the mapping being $1-1$ but not onto- That is, the solution to $A\mathbf{x} = \mathbf{b}$, if it exists, is unique (but it may not exist).

16. Suppose that A, B, C are $n \times n$ matrices with B invertible, and $I - BAB^{-1} = C$. Solve this equation for A. Be sure to show your work, and if you invert a matrix, explain why it is invertible.

SOLUTION:

$$I - BAB^{-1} = C \Rightarrow I - C = BAB^{-1} \Rightarrow B^{-1}I - B^{-1}C = AB^{-1} \Rightarrow B^{-1}B - B^{-1}CB = A \Rightarrow I - B^{-1}CB = A$$

Discussion Questions

1. If $T : X \to Y$ and $S : Y \to Z$ are linear transformations, show that the composition is a linear transformation from X to Z

SOLUTION (Also see the homework)- Show that $S(T(x_1 + x_2)) = S(T(x_1)) + S(T(x_2))$ and $S(T(cx)) = cS(T(x))$.

$$S(T(x_1 + x_2)) = S(T(x_1) + T(x_2)) = S(T(x_1)) + S(T(x_2))$$

and

$$S(T(cx)) = S(cT(x)) = cS(T(x))$$

2. Let A be $m \times n$. Let C be $n \times m$ so that $CA = I_n$. Show that the equation $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution. *NOTE: You cannot assume that* $n = m$*, because that might not be the case!*

SOLUTION: $A\mathbf{x} = \mathbf{0} \Rightarrow CA\mathbf{x} = 0 \Rightarrow \mathbf{x} = \mathbf{0}$

3. Let A be $m \times n$. Let D be an $n \times m$ matrix so that $AD = I_m$. Show that $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$.

SOLUTION: The solution is $\mathbf{x} = D\mathbf{b}$, since

$$A\mathbf{x} = A(D\mathbf{b}) = (AD)\mathbf{b} = I\mathbf{b} = \mathbf{b}$$

4. Show that if the columns of B are linearly dependent, then so are the columns of AB.

SOLUTION: A couple of versions of the solution are shown in the homework. Here's one way to do it:

If the columns of B are linearly dependent, then $B\mathbf{x} = \mathbf{0}$ has a non-trivial solution $\mathbf{x}$. Multiply by A, and

$$AB\mathbf{x} = \mathbf{0}$$

then has a non-trivial solution (the same non-trivial solution), so the columns of AB are linearly dependent.

5. Let A, B be $n \times n$. Show that if AB is invertible, then so is A.

Since AB is invertible, there is a matrix W so that $AB(W) = I$. Therefore, $A(BW) = I$, and if we let $C = BW$, there is a matrix C such that $AC = I$. Therefore, by the Invertible Matrix Theorem, A is invertible.

6. Let A be $n \times n$. If the equation $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution, do the columns of A span $\mathbb{R}^n$? Why or why not? Is your answer different if A is $n \times m$?

   SOLUTION: Yes. If the equation $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution, there is a pivot in every column. Since there are n columns, we have n pivots, so there is a pivot in every row, and therefore, $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$, and therefore the columns of A span $\mathbb{R}^n$.

   The answer is false if A is not square. Such a matrix would necessarily be tall, and the columns would not span the appropriate space.

7. Let T be a linear transformation. Show that if $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ are linearly dependent vectors, then $\{T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), T(\boldsymbol{v}_3)\}$ are linearly dependent vectors.

   SOLUTION: This is just like the homework (in the homework, we had two vectors). If the set of vectors is linearly dependent, there is a non-trivial solution to

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 = \mathbf{0}$$

   Apply T to both sides and use the linearity of T:

$$T(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3) = T(\mathbf{0}) \Rightarrow c_1T(\boldsymbol{v}_1) + c_2T(\boldsymbol{v}_2) + c_3T(\boldsymbol{v}_3) = \mathbf{0}$$

   This equation says that there is a non-trivial solution, so the vectors $\{T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), T(\boldsymbol{v}_3)\}$ are linearly dependent vectors.

8. If H is 7×7 matrix and $H\boldsymbol{x} = \boldsymbol{v}$ is consistent for every $\boldsymbol{v}$ in $\mathbb{R}^7$, then is it possible for $H\boldsymbol{x} = \boldsymbol{v}$ to have *more* than one solution for some $\boldsymbol{v} \in \mathbb{R}^7$? Why or why not?

   SOLUTION: If the given equation has a solution for every $\mathbf{v}$, then there is a pivot in every row of H. Since there are the same number of rows as columns, H also must have a pivot in every column, and so there can not be more than one solution for any $\mathbf{v}$.

9. Let A be invertible. Show that, if $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent, then $\{A\boldsymbol{v}_1, A\boldsymbol{v}_2, A\boldsymbol{v}_3\}$ is linearly independent. Is this true if A is not invertible?

   SOLUTION: A nice characterization of linearly independent column vectors is to say that, if $V = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3]$, then the columns are linearly independent if and only if the matrix equation $V\mathbf{c} = \mathbf{0}$ has only the trivial solution.

   The columns of AV are $\{A\boldsymbol{v}_1, A\boldsymbol{v}_2, A\boldsymbol{v}_3\}$ and we see that, if $AV\mathbf{c} = \mathbf{0}$ had a non-trivial solution and A is invertible, then apply the inverse to both sides to see that $V\mathbf{c} = \mathbf{0}$ has the same non-trivial solution. Therefore, $AV\mathbf{c} = \mathbf{0}$ must have only the trivial solution (and so the columns are linearly independent).

10. If A is $n \times n$ and invertible, show that A^T is invertible. (Hint: What should the inverse of A^T be? Show that your answer works). We show that the inverse will be $(A^{-1})^T$:

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$

True or False (and explain)?

1. If A is invertible, then the elementary row operations that reduce A to the identity also reduce A^{-1} to the identity.

   FALSE, because if A is row reduced to the identity, then

$$E_kE_{k-1}\cdots E_2E_1A = I$$

   Therefore, $A^{-1} = (E_kE_{k-1}\cdots E_2E_1)$.

   What you can say is the statement of Theorem 7, p. 123: The sequence of row ops that reduces A to I also transforms I into A^{-1}.

2. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each vector in $\mathbb{R}^n$ maps onto a unique vector in $\mathbb{R}^m$.

   FALSE, this is a statement of what it means to be a function. To be 1-1, each $\mathbf{y} \in \mathbb{R}^m$ must have come from a unique $\mathbf{x} \in \mathbb{R}^n$.

3. $A^T + B^T = (A+B)^T$ TRUE: This is a property of the transpose.

4. $(AB)C = (AC)B$ FALSE: The order of multiplication matters.

5. $((AB)^T)^{-1} = A^{-1}B^{-1}$ (you may assume A, B are invertible)

$$((AB)^T)^{-1} = (B^T A^T)^{-1} = (A^T)^{-1}(B^T)^{-1} = (A^{-1})^T (B^{-1})^T = (B^{-1}A^{-1})^T$$

   so FALSE.

6. If A is 5×5, and the columns of A do not span $\mathbb{R}^5$, it is possible that A is invertible.

   FALSE: There are several ways of describing what's happening here- For example,

   If the columns do not span $\mathbb{R}^5$, the mapping is not onto. Because A is square, the mapping is also not 1-1, and so the mapping is not invertible.

7. A linear transformation preserves the operations of vector addition and scalar multiplication.

   TRUE: This is another way to state the definition of linearity.

8. If $A\mathbf{x} = \mathbf{b}$ has more than 1 solution, so does $A\mathbf{x} = \mathbf{0}$.

   TRUE: True. Theorem 6 in Section 1.5 essentially says that when $A\mathbf{x} = \mathbf{b}$ is consistent, the solution sets of the non-homogeneous equation and the homogeneous equation are translates of each other. In this case, the two equations have the same number of solutions.

9. In some cases, it is possible for four vectors to span $\mathbb{R}^5$.

   FALSE: Construct the matrix A from the four vectors, so that A is 5×4. Then to say the vectors span $\mathbb{R}^5$ means that

$$A\mathbf{c} = \mathbf{b}$$

   has a solution for every $\mathbf{b}$. But we can have at most 4 pivots (the number of columns), so the last row in the rref of A must be a row of zeros, so the last column could be a pivot column for some choices of $\mathbf{b}$.

10. If A, B are row equivalent $m \times n$ matrices, and if the columns of A span $\mathbb{R}^m$, then so do the columns of B.

    True. If the columns of A span $\mathbb{R}^m$, then the reduced echelon form of A is a matrix U with a pivot in each row (For more info, see Theorem 4 in Section 1.4). Since B is row equivalent to A, B can be transformed by row operations first into A and then further transformed into U. Since U has a pivot in each row, so does B. Therefore, the columns of B span $\mathbb{R}^m$.