MATH 300, Second Exam REVIEW QUESTIONS

NOTE: You may use a calculator for this exam- You only need something that will perform basic arithmetic.

1. Let S be the parallelogram whose vertices are $(-1,1),(0,4),(1,2)$ and $(2,5)$. Use determinants to find the area of S.

   SOLUTION:

   If you plot the points, you should see which vectors we can take for the determinant:

   $$\mathbf{u} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

   Using these as the columns of the matrix A, the area is: $|1-6| = 5$.

2. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $B = \begin{bmatrix} a+2g & b+2h & c+2i \\ d+3g & e+3h & f+3i \\ g & h & i \end{bmatrix}$, and $C = \begin{bmatrix} g & h & i \\ 2d & 2e & 2f \\ a & b & c \end{bmatrix}$.

   If $\det(A) = 5$, find $\det(B), \det(C), \det(BC)$.

   SOLUTION:

   To get B from A, we take $2r_3 + r_1 \to r_1$ and $3r_3 + r_2 \to r_2$. These row operations do not change the determinant, so $\det(A) = \det(B) = 5$.

   To get the determinant of C, multiply the determinant of A by -2 (negative because of the row swap); we get -10.

   The determinant of BC is then $5 \cdot -10 = -50$

3. Assume that A and B are row equivalent, where:

   $$A = \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ -2 & -3 & 1 & -1 & -5 \\ -3 & -4 & 0 & -2 & -3 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 4 & 0 & -3 \\ 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) State which vector space contains each of the four subspaces, and state the dimension of each of the four subspaces:

   SOLUTION: Here are the spaces and dimensions:

   i. The row space of A is a subspace of $\mathbb{R}^5$ and has dimension 3 (this is the number of pivot rows).

   ii. The column space of A is a subspace of $\mathbb{R}^4$ and has dimension 3 (this is also known as the rank of A and is the number of pivot columns).

   iii. The null space of A is a subspace of $\mathbb{R}^5$ and has dimension 2 (the number of free variables).

   iv. The null space of A^T is a subspace of $\mathbb{R}^4$ and has dimension 1 (because the dimension of the column space is 3, and they should add to 4).

(b) Find a basis for Col(A): We see from the RREF that the pivot columns are cols 1, 2, and 4. Be sure to use the columns from the original matrix A! The basis is the set

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -4 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \\ 5 \end{bmatrix} \right\}$$

(c) Find a basis for Row(A): We can use the reduced rows (written as columns) for the basis:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -4 \end{bmatrix} \right\}$$

(d) Find a basis for Null(A): Solve $A\mathbf{x} = \mathbf{0}$, and the vectors are the basis vectors:

$$\begin{aligned} x_1 &= -4x_3 + 3x_5 \\ x_2 &= 3x_3 - 5x_5 \\ x_3 &= x_3 \\ x_4 &= \qquad 4x_5 \\ x_5 &= \qquad x_5 \end{aligned} \quad \Rightarrow \left\{ \begin{bmatrix} -4 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}$$

4. Determine if the following sets are subspaces of V. Justify your answers.

- $H = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, a \geq 0, \; b \geq 0, \; c \geq 0 \right\}, \qquad V = \mathbb{R}^3$

  SOLUTION: This set is not closed under scalar multiplication (for example, if $c = -1$).

- $H = \left\{ \begin{bmatrix} a + 3b \\ a - b \\ 2a + b \\ 4a \end{bmatrix}, \; a, b \text{ in } \mathbb{R} \right\}, \quad V = \mathbb{R}^4$

  SOLUTION: This is the span of $[1, 1, 2, 4]^T$ and $[3, -1, 1, 0]^T$, so H is a subspace.

- $H = \{f : f'(x) = f(x)\}, V = C^1[\mathbb{R}]$

  (C^1 is the space of differentiable functions where the derivative is continuous).

  SOLUTION: We could show this directly:

  - The zero function is in H since the derivative of 0 is 0.
  - If u, v are in H, then $u' = u$ and $v' = v$. Therefore, $(u + v)' = u' + v' = u + v$, so $u + v$ is in H.
  - If $u \in H$, then $u' = u$. Therefore, $(cu)' = cu' = cu$, so cu is in H.

- H is the set of vectors in $\mathbb{R}^3$ whose first entry is the sum of the second and third entries, $V = \mathbb{R}^3$.

SOLUTION: Rewriting H algebraically, it is the set of vectors in $\mathbb{R}^3$ so that

$$\begin{bmatrix} a+b \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Since H is the span of two vectors, it is a subspace.

5. Prove that, if $T : V \mapsto W$ is a linear transformation between vector spaces V and W, then the range of T, which we denote as $T(V)$, is a subspace of W.

   SOLUTION: Show that the three parts to the definition hold true-

   - Since $\mathbf{0} \in V$ (V is a subspace) and $T(\mathbf{0}) = \mathbf{0}$, then $\mathbf{0} \in T(V)$.
   - Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be in $T(V)$. Show that the sum is in $T(V)$.
     Since $\mathbf{w}_1, \mathbf{w}_2 \in T(V)$, there are vectors, $\mathbf{v}_1, \mathbf{v}_2$ in V such that $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$. Now by linearity, we have:

     $$\mathbf{w}_1 + \mathbf{w}_2 = T(\mathbf{v}_1) + T(\mathbf{v}_2) = T(\mathbf{v}_1 + \mathbf{v}_2)$$

     Therefore, $\mathbf{w}_1 + \mathbf{w}_2$ is the image of $\mathbf{v}_1 + \mathbf{v}_2$ which is in V (because V is a subspace). Therefore, $\mathbf{w}_1 + \mathbf{w}_2$ is in $T(V)$.
   - Let $\mathbf{w} \in T(V)$, and show $c\mathbf{w} \in T(V)$.
     Since $\mathbf{w} \in T(V)$, there exists $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{w}$. We also know that $c\mathbf{v} \in V$ because V is a subspace. By the linearity of T,

     $$c\mathbf{w} = cT(\mathbf{v}) = T(c\mathbf{v})$$

     and so $c\mathbf{w} \in T(V)$.

6. Let H, K be subspaces of vector space V. Define $H + K$ as the set below:

   $$H + K = \{\mathbf{w} \mid \mathbf{w} = \mathbf{u} + \mathbf{v}, \text{ for some } \mathbf{u} \in H, \mathbf{v} \in K\}$$

   SOLUTION:

   - Since $\mathbf{0} \in H$ and K, and $\mathbf{0} + \mathbf{0} = \mathbf{0}$, then $\mathbf{0} \in H + K$
   - Let $\mathbf{w}_1, \mathbf{w}_2 \in H + K$. Show that the sum is in $H + K$:
     Since $\mathbf{w}_1, \mathbf{w}_2 \in H + K$, there are vectors, $\mathbf{u}_1, \mathbf{u}_2$ in H and vectors, $\mathbf{v}_1, \mathbf{v}_2$ in K such that

     $$\begin{array}{ll} \mathbf{w}_1 & = \mathbf{u}_1 + \mathbf{v}_1 \\ \mathbf{w}_2 & = \mathbf{u}_2 + \mathbf{v}_2 \end{array} \Rightarrow \mathbf{w}_1 + \mathbf{w}_2 = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{v}_1 + \mathbf{v}_2)$$

     with $\mathbf{u}_1 + \mathbf{u}_2 \in H$ and $\mathbf{v}_1 + \mathbf{v}_2 \in K$ because H, K are subspaces.
   - Let $\mathbf{w} \in H + K$. Show that $c\mathbf{w} \in H + K$:
     If $\mathbf{w} \in H + K$, then there are vectors $\mathbf{u} \in H$ and $\mathbf{v} \in K$ so that $\mathbf{w} = \mathbf{u} + \mathbf{v}$. Therefore,

     $$c\mathbf{w} = c\mathbf{u} + c\mathbf{v}$$

     and since H and K are subspaces, $c\mathbf{u} \in H$ and $c\mathbf{v} \in K$. Therefore, $c\mathbf{w}$ is a sum of something in H and something in K, so it is in $H + K$.

7. Let A be an $n \times n$ matrix. Write statements from the Invertible Matrix Theorem that are each equivalent to the statement "A is invertible". Use the following concepts, one in each statement: (a) Null(A) (b) Basis (c) Rank

   SOLUTION:

   (a) $\text{Null}(A) = \{0\}$ (b) The columns of A form a basis for $\mathbb{R}^n$ (c) The rank of A is n.

8. Is it possible that all solutions of a homogeneous system of ten linear equations in twelve variables are multiples of one fixed nonzero solution? Discuss.

   SOLUTION:

   No. If we have 10 equations in 12 variables, we must have at least two free variables, and so the null space of the corresponding matrix is the span of at least two linearly independent (nonzero) vectors.

9. Show that $\{1, 2t, -2 + 4t^2\}$ is a basis for P_2.

   SOLUTION: Using the isomorphism theorem to connect P_2 to $\mathbb{R}^3$, we look to see if the images of the candidate vectors are linearly independent. If they are, since P_2 is three dimensional and we have three linearly independent vectors, then they must form a basis for P_2.

   To convert the vectors into their corresponding coordinates (using the standard basis for P_2):

$$1 \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad 2t \mapsto \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \qquad -2 + 4t^2 \mapsto \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$$

   These are clearly linearly independent (upper triangular). If it wasn't clear, form a matrix using these vectors as the columns, and do row reduction. If they are linearly independent, there should be a pivot in every column.

10. Let $T : V \to W$ be a $1-1$ and linear transformation on vector space V to vector space W. Show that if $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ are linearly dependent vectors in W, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly dependent vectors in V.

    SOLUTION:

    If $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ are linearly dependent, then there is a nontrivial solution to

$$c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3) = 0$$

    Since T is linear, we can write this equation as:

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = T(0)$$

    If T is $1-1$, we can say that $T(a) = T(b)$ implies that $a = b$. In this case, we then can say that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

    So that we again get a non-trivial solution, and these vectors are linearly dependent.

11. Use Cramer's Rule to solve the system:

$$\begin{aligned} 2x_1 + x_2 \quad &= 7 \\ -3x_1 + x_3 \quad &= -8 \\ x_2 + 2x_3 \quad &= -3 \end{aligned}$$

SOLUTION: We'll need some determinants:

$$\begin{vmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 4 \qquad \begin{vmatrix} 7 & 1 & 0 \\ -8 & 0 & 1 \\ -3 & 1 & 2 \end{vmatrix} = 6 \qquad \begin{vmatrix} 2 & 7 & 0 \\ -3 & -8 & 1 \\ 0 & -3 & 2 \end{vmatrix} = 16 \qquad \begin{vmatrix} 2 & 1 & 7 \\ -3 & 0 & -8 \\ 0 & 1 & -3 \end{vmatrix} = -14$$

So the solution is:

$$x_1 = \frac{6}{4} = \frac{3}{2} \qquad x_2 = \frac{16}{4} = 4 \qquad x_3 = \frac{-14}{4} = \frac{-7}{2}$$

12. Let $A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$, and $\mathbf{w} = [2, 1]^T$. Is $\mathbf{w}$ in the column space of A? Is it in the null space of A?

SOLUTION: The first question is asking if $\mathbf{w}$ is in the span of the columns of A. By inspection, the weights are $-1/3$ and 0, so yes.

The second question is asking if $A\mathbf{w} = \mathbf{0}$. In this case, it is true as well.

Side Note: It was only a coincidence that this vector was in both the column space as well as the null space. If A was $m \times n$, then the column space would be in $\mathbb{R}^m$ and the null space would be in $\mathbb{R}^n$, so these would definitely be different.

13. Prove that the column space is a vector space using a very short proof, then prove it directly by showing the three conditions hold.

SOLUTION:

For the short proof, we know that any spanning set is automatically a subspace. The column space is the span of the columns of A.

Now for the longer proof using the definition:

Recall that a vector $\mathbf{b}$ is in the column space of A if and only if $\mathbf{b} = A\mathbf{x}$ for some $\mathbf{x}$.

(a) Since $A\mathbf{0} = \mathbf{0}$, $\mathbf{0} \in \text{Col}(A)$.

(b) Let $\mathbf{b}_1, \mathbf{b}_2 \in \text{Col}(A)$. Then there exists $\mathbf{x}_1, \mathbf{x}_2$ so that

$$A(\mathbf{x}_1) = \mathbf{b}_1 \qquad A(\mathbf{x}_2) = \mathbf{b}_2 \text{ therefore } \mathbf{b}_1 + \mathbf{b}_2 = A(\mathbf{x}_1) + A(\mathbf{x}_2) = A(\mathbf{x}_1 + \mathbf{x}_2)$$

This shows that $\mathbf{b}_1 + \mathbf{b}_2$ is in the column space of A.

(c) Let $\mathbf{b} \in \text{Col}(A)$. Then $A\mathbf{x} = \mathbf{b}$ for some $\mathbf{x}$ in the domain, and

$$c\mathbf{b} = cA\mathbf{x} = A(c\mathbf{x})$$

Therefore, $c\mathbf{b}$ is in the column space of A for all constants c.

14. If A, B are 4×4 matrices with $\det(A) = 2$ and $\det(B) = -3$, what is the determinant of the following (if you can compute it):

SOLUTIONS:

(a) $\det(AB) = -6$, (b) $\det(A^{-1}) = 1/2$, (c) $\det(5B) = 5^3(-3)$, (d) $\det(3A - 2B)$ is unknown with what is given, (e) $\det(B^T) = -3$

15. True or False, and give a short reason:

(a) If $\det(A) = 2$ and $\det(B) = 3$, then $\det(A + B) = 5$.
SOLUTION: False. $\det(A + B) \neq \det(A) + \det(B)$.

(b) Let A be $n \times n$. Then $\det(A^T A) \geq 0$.
SOLUTION: True: $\det(A^T A) = \det(A^T)\det(A) = (\det(A))^2 \geq 0$.

(c) If A^3 is the zero matrix, then $\det(A) = 0$.
SOLUTION: True- If $A^3 = 0$, then $0 = \det(A^3) = (\det(A))^3$, so $\det(A) = 0$.

(d) $\mathbb{R}^2$ is a two dimensional subspace of $\mathbb{R}^3$.
SOLUTION: False. $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$, they are completely different vector spaces. However, a two dimensional subspace in $\mathbb{R}^3$ is isomorphic to $\mathbb{R}^2$.

(e) Row operations preserve the linear dependence relations among the rows of A.
SOLUTION: False. Swapping rows will especially change the relations- However, row operations do NOT change the linear dependence relations among the **columns** of A.

(f) The sum of the dimensions of the row space and the null space of A equals the number of rows of A.
SOLUTION: False. The sum is the number of *columns* of A.

16. Let the matrix A and its RREF, R_A, be given as below:

$$A = \begin{bmatrix} 1 & 1 & 7 & 2 & 2 \\ 3 & 0 & 9 & 3 & 4 \\ -3 & 1 & -5 & -2 & 3 \\ 2 & 2 & 14 & 4 & 2 \end{bmatrix} \qquad R_A = \begin{bmatrix} 1 & 0 & 3 & 1 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so that the columns of A are $\mathbf{a}_1, \cdots, \mathbf{a}_5$.

Similarly, define Z and its RREF, R_Z, as:

$$Z = \begin{bmatrix} 4 & 5 & 3 & 4 \\ 5 & 6 & 5 & -3 \\ 10 & -3 & 9 & -106 \\ 4 & 10 & 2 & 44 \end{bmatrix} \qquad R_z = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Label the columns of Z as $\mathbf{z}_1, \cdots, \mathbf{z}_5$.

(a) Find the rank of A and a basis for the column space of A (use the notation $\mathbf{a}_1$, etc.). Similarly, do the same for Z:
SOLUTION:

The rank is the dimension of the column space (the number of pivot columns in the RREF), so in this case it is 3. The corresponding columns of A are $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5$ and these form a basis for Col(A).

Similarly, the rank of Z is 3 and a basis for Col(Z) is $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$.

(b) You'll notice that the rank of A is the rank of Z. Here is a row reduction using some columns of A and Z:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 4 & 5 & 3 \\ 3 & 0 & 4 & 5 & 6 & 5 \\ -3 & 1 & 3 & 10 & -3 & 9 \\ 2 & 2 & 2 & 4 & 10 & 2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Are the subspaces spanned by the columns of A and Z equal?

SOLUTION: The RREF tells us that $\mathbf{z}_1, \mathbf{z}_2$ and $\mathbf{z}_3$ can be written in terms of $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_5$. The subspaces are both three dimensional, so if we have any set of 3 linearly independent vectors (in the subspace) we must have a basis (therefore, the basis for Col(Z) could be used for Col(A) and vice-versa).

(c) Let $\mathcal{B}$ and be the set of basis vectors used for the column spaces of A found in (a). Find the change of coordinates matrix $P_{\mathcal{B}}$ that changes the coordinates from $\mathcal{B}$ to the standard basis, then find the coordinates of $\mathbf{z}_1$ with respect to $\mathcal{B}$ (Hint: The second part does not rely on the first).

SOLUTION: The matrix is formed from the columns of A:

$$P_{\mathcal{B}} = [\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_5]$$

The coordinates of $\mathbf{z}_1$ are shown in the row reduction in part (b):

$$\mathbf{z}_1 = -\mathbf{a}_1 + \mathbf{a}_2 + 2\mathbf{a}_5$$

so the coordinate vector is $[-1, 1, 2]^T$.

(d) Find the coordinates of $\mathbf{z}_4$ using the basis vectors in $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$.

SOLUTION: From the initial setup, we see that

$$[\mathbf{z}_4]_C = \begin{bmatrix} -4 \\ 7 \\ -5 \end{bmatrix}$$

17. Short Answer:

(a) Define the *kernel* of a transformation T:

SOLUTION: The kernel is the set of all x such that $T(x) = 0$.

(b) Define the *dimension* of a vector space:

SOLUTION: The dimension is the number of basis vectors for the vector space, with the special cases of dimension 0 for the vector space containing only the zero vector, and if the vector space cannot be spanned by a finite set, it is infinite dimensional.

(c) We said that $\mathbb{P}_n$ is isomorphic to $\mathbb{R}^{n+1}$. What is the isomorphism?

SOLUTION: The isomorphism is the coordinate mapping. That is:

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n \in \mathbb{P}_n \mapsto [a_0, a_1, \cdots, a_n]^T \in \mathbb{R}^{n+1}$$

(d) If C is 4×5, what is the largest possible rank of C?

SOLUTION: The largest possible rank is 4. (If C has its largest possible rank, it is said to be full rank).

What is the smallest possible dimension of the null space of C?

SOLUTION: The dimension of the null space is the number of free variables in the RREF of the matrix (or the number of non-pivot columns). If the rank is 4, the dimension of the null space must be at 1. (Note: If the rank were smaller than 4, the extra dimensions are added to the null space).

(e) If A is a 4×7 matrix with rank 3, find the dimensions of the four fundamental subspaces of A.

SOLUTION: If A has rank 3, then the dimension of the row space is 3, the dimension of the null space is 4, the dimension of the column space is 3, and the dimension of $\text{Null}(A^T)$ is 1.

(f) Show that the coordinate mapping (from $n-$dimensional vector space V to $\mathbb{R}^n$) is onto.

SOLUTION: Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a basis for V. Then for any vector $\mathbf{c}$ in $\mathbb{R}^n$, the associated vector in V is given by:

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$$

which is in V because V is a vector space. Therefore, the coordinate mapping is *onto*.

18. Let A be $m \times n$ and let B be $n \times p$. Show that the $\text{rank}(AB) \leq \text{rank}(A)$. (Hint: Explain why every vector in the column space of AB is in the column space of A).

SOLUTION: Using the hint, if $\mathbf{y} \in \text{Col}(AB)$, then $\mathbf{y} = AB\mathbf{x}$ for some $\mathbf{x}$. Therefore, $\mathbf{y} = A(B\mathbf{x}) = A\mathbf{w}$, so $\mathbf{y}$ must be in the column space of A. Therefore, the column space of AB is a subspace of the column space of A, so

$$\dim(\text{Col}(AB)) \leq \dim(\text{Col}(A))$$

which gives the result, since the rank is the dimension of the column space.

19. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

(a) If T is one-to-one, what is the dimension of the range of T?

SOLUTION: If T is $1-1$, then it is invertible on its range. Therefore, it will map the basis vectors of $\mathbb{R}^n$ to linearly independent vectors in $\mathbb{R}^m$. These will form the basis for the range, so the range is n dimensional.

Alternative: Since $T : \mathbb{R}^n \to \mathbb{R}^m$, there is an $m \times n$ matrix A so that $A\mathbf{x} = T(\mathbf{x})$. If T is $1-1$, the solution to $A\mathbf{x} = 0$ is only the trivial solution, so the dimension of the null space is 0, and therefore, the dimension of the column space is n (and that is the range of T).

(b) What is the dimension of the kernel of T if T maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$? Explain.

SOLUTION: Let A be the $m \times n$ matrix for the linear map. If the column space of A has dimension m, then so does the row space, and so the dimension of the null space would be $n - m$, which is valid only if $n \geq m$ (wide or square matrix).

20. Find the determinant of the matrix A below:

$$A = \begin{bmatrix} 4 & 8 & 8 & 8 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 6 & 8 & 8 & 8 & 7 \\ 0 & 8 & 8 & 3 & 0 \\ 0 & 8 & 2 & 0 & 0 \end{bmatrix}$$

SOLUTION: Expand in terms of the second row, so the determinant is the determinant of the smaller and smaller matrices:

$$\begin{vmatrix} 4 & 8 & 8 & 8 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 6 & 8 & 8 & 8 & 7 \\ 0 & 8 & 8 & 3 & 0 \\ 0 & 8 & 2 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 4 & 8 & 8 & 5 \\ 6 & 8 & 8 & 7 \\ 0 & 8 & 3 & 0 \\ 0 & 2 & 0 & 0 \end{vmatrix} = 2 \begin{vmatrix} 4 & 8 & 5 \\ 6 & 8 & 7 \\ 0 & 3 & 0 \end{vmatrix} = 2 \cdot (-3) \begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix} = 12$$