Example Questions, Exam 3, Math 300

1. (Corrected Copy) Let A and its RREF be given as:

$$A = \begin{bmatrix} -1 & -5 & 3 & 9 \\ -48 & -40 & 24 & 92 \\ 94 & 70 & -42 & -166 \\ -48 & -40 & 24 & 92 \end{bmatrix} \qquad \text{rref}(A) = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 0 & 10 & -3 & -17 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We also note two facts: $\lambda = 4$ is an eigenvalue of A, and $\mathbf{u} = [1, 0, 2, 0]^T$ is an eigenvector of A.

(a) Find a basis for the eigenspace E_4:

(b) What is the eigenvalue for the eigenvector $\mathbf{u}$?

(c) You might have noticed that the second and fourth rows are the same. Does that imply we have a certain eigenvalue? Find a basis for its eigenspace. To save you some time, we have included the RREF of A.

(d) What is the characteristic polynomial of A?

(e) Show that A is diagonalizable by finding an appropriate P and D.

2. Short Answer:

(a) Show that if A^2 is the zero matrix, the only eigenvalue of A is zero.

(b) (You may use a calculator) Consider $\frac{1-3i}{2+i}$.

- Write the complex number in $a + ib$ form.
- Write the complex number in polar form: $re^{i\theta}$.

(c) Normalize the vector $[1, -2, 1, 1]^T$.

(d) Suppose A is 3×3, and $\mathbf{u}$ is an eigenvector of A corresponding to an eigenvalue of 7.
Is $\mathbf{u}$ an eigenvector of $2I - A$? If so, find the corresponding eigenvalue. If not, explain why not.

(e) True or False? A matrix with orthonormal columns is an orthogonal matrix.

3. Show the following: If U, V are orthogonal matrices, then so is UV.

4. Let $A = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$.

(a) Is A a regular stochastic matrix?

(b) Diagonalize the matrix A.

(c) Use the diagonalization to find a product for A^k.

(d) What happens to A^k as $k \to \infty$?

5. Let U be $m \times n$ with orthonormal columns. Show that the length of $U\mathbf{x}$ is the same as the length of $\mathbf{x}$. Use the first part of your answer to show that the angle between $\mathbf{x}$ and $\mathbf{y}$ is the same as the angle between $U\mathbf{x}$ and $U\mathbf{y}$.

6. Show that the eigenvalues of A and A^T are the same.

7. (Corrected Copy) Show that if $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are eigenvectors corresponding to distinct eigenvalues, then the vectors are linearly independent.

8. Find the eigenvalues and bases for the eigenspaces if $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$.

9. Compute an appropriate factorization for the matrix $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$.

10. Let matrix A be $m \times n$. Show that the row space is orthogonal to the null space.

11. If $\mathbf{u} = [3, 2, -5, 0]$ and $\mathbf{v} = [1, 1, -1, 2]$, then compute:

    (a) The distance between $\mathbf{u}$ and $\mathbf{v}$.

    (b) An approximate angle between $\mathbf{u}$ and $\mathbf{v}$ (use your calculator).

    (c) The orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$

12. Prove the Pythagorean Theorem for two vectors $\mathbf{x}$ and $\mathbf{y}$:

$$\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

13. On a given day, a student is either healthy or ill. Of the students that are healthy today, 95% will be healthy tomorrow. Of the students that are ill today, 55% will still be ill tomorrow.

    (a) What is the stochastic matrix for this situation?

    (b) If 20% of the students are ill on Monday, what percentage are likely to be ill on Wednesday?

14. If each row of A sums to the same number s, what is one eigenvalue and eigenvector? If each column of A sums to the same number s, does your previous answer hold?

15. If A is similar to B, show that they have the same eigenvalues.

16. Prove that if the set $\{\mathbf{v}_1, \cdots \mathbf{v}_k\}$ form a basis for subspace W, and $\mathbf{x}$ is orthogonal to each $\mathbf{v}_i$, for $i = 1$ to k, then $\mathbf{x}$ is orthogonal to W. (Hint: Start with a generic vector $\mathbf{w} \in W$, and show that $\mathbf{x} \cdot \mathbf{w} = 0$.)

17. Prove that the eigenvalues of a triangular matrix are the entries on its main diagonal.

18. (More) True or False? If the statement is false and you can provide a counterexample to demonstrate this, then do so. If the statement is false and be can slightly modified so as to make it true then indicate how this may be done.

    (a) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent eigenvectors, then they correspond to distinct eigenvalues.

    (b) If A is invertible, then A is diagonalizable.

    (c) The orthogonal projection of $\mathbf{y}$ onto a vector $\mathbf{v}$ is the same as the orthogonal projection of $\mathbf{y}$ onto $c\mathbf{v}$ whenever $c \neq 0$.

    (d) If A is an orthogonal matrix, then A^T is an orthogonal matrix.

    (e) If A, B have the same eigenvalues, then they are similar.