Example Solutions, Exam 3, Math 300

1. **TYPO: A row reduced version of A is corrected below** Let A and its RREF be given as:

$$A = \begin{bmatrix} -1 & -5 & 3 & 9 \\ -48 & -40 & 24 & 92 \\ 94 & 70 & -42 & -166 \\ -48 & -40 & 24 & 92 \end{bmatrix} \qquad \text{rref}(A) = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 0 & 10 & -3 & -17 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We also note two facts: $\lambda = 4$ is an eigenvalue of A, and $\mathbf{u} = [1, 0, 2, 0]^T$ is an eigenvector of A.

(a) Find a basis for the eigenspace E_4:
SOLN: We find that $A - 4I$ row reduces to the following, which means it is only 1-dimensional:

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

(b) What is the eigenvalue for the eigenvector $\mathbf{u}$?
SOLN: We see that $A\mathbf{u}$ is the sum of the first and twice the third column of A:

$$A\mathbf{u} = \begin{bmatrix} 5 \\ 0 \\ 10 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \lambda = 5$$

(c) You might have noticed that the second and fourth rows are the same. Does that imply we have a certain eigenvalue? Find a basis for its eigenspace. To save you some time, we have included the RREF of A.
SOLN: If rows 2 and 4 are the same, the matrix A is not invertible, and $\lambda = 0$ is an eigenvalue. Using the given RREF,

$$\begin{array}{ll} v_1 & = \quad 1/2 v_4 \\ v_2 & = 3/10 v_3 + 17/10 v_4 \\ v_3 & = v_3 \\ v_4 & = \quad v_4 \end{array} \quad \Rightarrow \quad E_0 = \text{span}\left\{ \begin{bmatrix} 0 \\ 3 \\ 10 \\ 0 \end{bmatrix} \begin{bmatrix} 5 \\ 17 \\ 0 \\ 10 \end{bmatrix} \right\}$$

NOTE: We can check our answer since these vectors, which form a basis for the null space of A, should be orthogonal to the row space of A, whose basis is formed from the reduced form.

(d) What is the characteristic polynomial of A?
SOLN: We know that the eigenvalues are $4, 5, 0, 0$, so the polynomial in factored form is:

$$\lambda^2(\lambda - 4)(\lambda - 5)$$

(e) Show that A is diagonalizable by finding an appropriate P and D.
SOLN: We have already found all the components,

$$P = \begin{bmatrix} 2 & 5 & 0 & 5 \\ 1 & 0 & 3 & 17 \\ 2 & 10 & 10 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Short Answer:

(a) Show that if A^2 is the zero matrix, the only eigenvalue of A is zero.

SOLN: If $A\mathbf{v} = \lambda\mathbf{v}$, then $A^2\mathbf{v} = \lambda A\mathbf{v} = \lambda^2\mathbf{v}$. Therefore, if A^2 is the zero matrix, then $\lambda^2 = 0$ (since $\mathbf{v}$ cannot be the zero vector), or λ must be zero.

(b) (You may use a calculator) Consider $\frac{1-3i}{2+i}$.

- Write the complex number in $a + ib$ form.
  SOLN: $\frac{-1}{5} - \frac{7}{5}i$
- Write the complex number in polar form: $re^{i\theta}$.
  SOLN: $r = \sqrt{2}$ and $\theta = tan^{-1}(7) + \pi \approx -1.713$

(c) Normalize the vector $[1, -2, 1, 1]^T$.

SOLN: Multiply by the reciprocal of the length, the length being $\sqrt{1+4+1+1} = \sqrt{7}$, so the new vector is: $[1/\sqrt{7}, -2/\sqrt{7}, 1/\sqrt{7}, 1/\sqrt{7}]^T$.

(d) Suppose A is 3×3, and $\mathbf{u}$ is an eigenvector of A corresponding to an eigenvalue of 7.

Is $\mathbf{u}$ an eigenvector of $2I - A$? If so, find the corresponding eigenvalue. If not, explain why not.

SOLN: Let's see:

$$(2I - A)\mathbf{u} = 2\mathbf{u} - A\mathbf{u} = 2\mathbf{u} - 7\mathbf{u} = -5\mathbf{u}$$

So yes, $\mathbf{u}$ is an eigenvector, with the new eigenvalue -5.

(e) True or False? A matrix with orthonormal columns is an orthogonal matrix.

SOLN: False- By definition, an orthogonal matrix must be a *square* matrix with orthonormal columns.

3. Show the following: If U, V are orthogonal matrices, then so is UV.

SOLN: To show that a matrix B is orthogonal, show that $B^TB = I$. In this case, let $B = UV$, so:

$$B^TB = (UV)^T(UV) = V^TU^TUV = V^TIV = V^TV = I$$

The third equality is due to U being orthogonal, as is the last equality.

4. Let $A = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$.

(a) Is A a regular stochastic matrix?

SOLUTION: Yes, A is a regular stochastic matrix (it is stochastic because all entries are non-negative, and the columns sum to 1. It is regular because there are no zero entries).

(b) Diagonalize the matrix A.

SOLUTION: To diagonalize the matrix, find the eigenvalues and eigenvectors. Since A is stochastic, $\lambda = 1$ is always an eigenvalue. For the other, use the quadratic formula to get $\lambda = 1/2$.

For $\lambda = 1$, solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$. Do try to make your life a bit easier by the following simplifications:

$$\begin{bmatrix} -0.3 & 0.2 \\ 0.3 & -0.2 \end{bmatrix} \rightarrow \begin{bmatrix} -3 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Similarly, for $\lambda = 1/2$, we have:

$$\begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Therefore,

$$A = PDP^{-1} \text{ where } P = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \quad P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix}$$

(For a 2×2, go ahead and give P^{-1}).

(c) Use the diagonalization to find a product for A^k.
SOLUTION:

$$A^k = P \begin{bmatrix} 1 & 0 \\ 0 & 0.5^k \end{bmatrix} P^{-1}$$

where P, P^{-1} are as given in the previous answer.

(d) What happens to A^k as $k \to \infty$?
SOLUTION: As $k \to \infty$,

$$A^k \to P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1} = \frac{1}{5} \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$$

5. Let U be $m \times n$ with orthonormal columns. Show that the length of $U\mathbf{x}$ is the same as the length of $\mathbf{x}$. Use the first part of your answer to show that the angle between $\mathbf{x}$ and $\mathbf{y}$ is the same as the angle between $U\mathbf{x}$ and $U\mathbf{y}$.

SOLUTION: For the first question, we show that $\|U\mathbf{x}\|^2 = \|\mathbf{x}\|^2$ (which is equivalent, but does away with the square root). Start with the dot product definition of the norm:

$$\|U\mathbf{x}\|^2 = (U\mathbf{x}) \cdot (U\mathbf{x}) = (U\mathbf{x})^T U\mathbf{x} = \mathbf{x}^T U^T U\mathbf{x} = \mathbf{x}^T I\mathbf{x} = \mathbf{x}^T\mathbf{x} = \|\mathbf{x}\|^2$$

The angle between $\mathbf{x}$ and $\mathbf{y}$ is given by solving:

$$\cos(\theta) = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|}$$

In the denominator, we know from what we just computed that $\|U\mathbf{x}\| = \|\mathbf{x}\|$. Therefore, the statement will be true if $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$:

$$(U\mathbf{x}) \cdot (U\mathbf{y}) = (U\mathbf{x})^T (U\mathbf{y}) = \mathbf{x}^T U^T U\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$$

Therefore,

$$\frac{(U\mathbf{x}) \cdot (U\mathbf{y})}{\|U\mathbf{x}\|\,\|U\mathbf{y}\|} = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|}$$

6. Show that the eigenvalues of A and A^T are the same.

SOLUTION: Begin with λ being an eigenvalue of A. We show that $|A^T - \lambda I| = 0$. Since λ is an eigenvalue of A, and the determinant of the transpose is the determinant of the matrix, we have:

$$0 = |A - \lambda I| = |(A - \lambda I)^T| = |A^T - \lambda I^T| = |A^T - \lambda I|$$

(the last equality is because I is symmetric- $I^T = I$). Therefore, λ is an eigenvalue of A^T.

7. (Corrected to be a set of n vectors and a hint given in class) Show that if $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are eigenvectors corresponding to distinct eigenvalues, then the vectors are linearly independent. **Hint (given in class):** You may suppose that the eigenvectors are linearly dependent. Then there is a $p \le n$ so that

$$\mathbf{v}_p = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{p-1}v_{p-1}$$

If we multiply both sides by A, we get:

$$A\mathbf{v}_p = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + \cdots + c_{p-1}Av_{p-1} = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + \cdots + c_{p-1}\lambda_{p-1}\mathbf{v}_{p-1}$$

If we multiply both sides by λ_p, we get:

$$\lambda_p\mathbf{v}_p = c_1\lambda_p\mathbf{v}_1 + c_2\lambda_p\mathbf{v}_2 + \cdots + c_{p-1}\lambda_p\mathbf{v}_{p-1}$$

Subtract these and we get:

$$0 = c_1(\lambda_1 - \lambda_p)\mathbf{v}_1 + \cdots + c_{p-1}(\lambda_{p-1} - \lambda_p)\mathbf{v}_{p-1}$$

These are linearly independent, so that means $\lambda_i - \lambda_p$ must all be zero. But we said that the eigenvalues are distinct (so we get a contradiction). Therefore, the statement we started with must be false: The eigenvectors must be linearly independent.

NOTE: The main thing I want you to see here is how we can expand a vector in terms of eigenvectors, then multiply by A and analyze the result. The proof we gave is called a "proof by contradiction"- I won't ask you to do the whole thing on the exam without some hints along the way, as we did here.

8. Find the eigenvalues and bases for the eigenspaces if $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$.

   SOLUTION: In this case, the eigenvalues are complex. We give one here- The others are the conjugates.

$$\lambda = 3 - i \qquad \mathbf{v} = \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$$

   NOTE: If you get a different $\mathbf{v}$, see if you can determine if yours is a constant multiple (could be a complex multiple) of mine.

9. Compute an appropriate factorization for the matrix $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$.

   SOLUTION: Use the real and imaginary parts of the vector $\mathbf{v}$ to construct the matrix P:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$$

10. Let matrix A be $m \times n$. Show that the row space is orthogonal to the null space.

    SOLUTION: Since the rows of A span the row space, it will suffice to show that each row vector is orthogonal to any vector in the null space. Let $\mathbf{x}$ be any arbitrary vector in the null space of A. Then

$$A\mathbf{x} = \mathbf{0}$$

    Now write A in terms of its rows:

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} A\mathbf{x} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix} \mathbf{x} = \begin{bmatrix} r_1 \cdot \mathbf{x} \\ r_2 \cdot \mathbf{x} \\ \vdots \\ r_m \cdot \mathbf{x} \end{bmatrix}$$

    Therefore, the dot product of any row of A with $\mathbf{x}$ must be zero, and so the rows of A are orthogonal to the null space of A.

11. If $\mathbf{u} = [3, 2, -5, 0]$ and $\mathbf{v} = [1, 1, -1, 2]$, then compute:

    (a) The distance between $\mathbf{u}$ and $\mathbf{v}$.
    SOLUTION: The distance between $\mathbf{u}$ and $\mathbf{v}$ is computed by $\|\mathbf{u}-\mathbf{v}\| = \sqrt{2^2 + 1^2 + (-4)^2 + (-2)^2} = 5$

    (b) An approximate angle between $\mathbf{u}$ and $\mathbf{v}$ (use your calculator).
    SOLUTION:

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{10}{\sqrt{38}\sqrt{7}} \approx 0.6131$$

    Therefore, $\theta = \cos^{-1}(0.6131) \approx 0.9108$ radians, or 52.2 degrees.

(c) The orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$
SOLUTION:

$$\text{Proj}_v(\mathbf{u}) = \frac{\mathbf{u}^T\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{10}{7}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}$$

12. Prove the Pythagorean Theorem for two vectors $\mathbf{x}$ and $\mathbf{y}$:

$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

NOTE: The vectors $\mathbf{x}, \mathbf{y}$ must be orthogonal for this to be true, which we will see. First, write the norm in terms of the dot product, then expand and simplify:

$$\|\mathbf{x}+\mathbf{y}\|^2 = (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) = (\mathbf{x}+\mathbf{y})^T(\mathbf{x}+\mathbf{y}) = (\mathbf{x}^T+\mathbf{y}^T)(\mathbf{x}+\mathbf{y}) = \mathbf{x}^T\mathbf{x} + 2\mathbf{x}^T\mathbf{y} + \mathbf{y}^T\mathbf{y}$$

We get the desired result if (and only if) the vectors $\mathbf{x}$, $\mathbf{y}$ are orthogonal.

13. On a given day, a student is either healthy or ill. Of the students that are healthy today, 95% will be healthy tomorrow. Of the students that are ill today, 55% will still be ill tomorrow.

(a) What is the stochastic matrix for this situation?
SOLUTION: If our state vector is in the order [healthy, sick]^T then the stochastic matrix is:

$$A = \begin{bmatrix} 0.95 & 0.45 \\ 0.05 & 0.55 \end{bmatrix}$$

(b) If 20% of the students are ill on Monday, what percentage are likely to be ill on Wednesday?
SOLUTION: Compute $A^2\mathbf{x}$, where $\mathbf{x} = [0.8, 0.2]^T$. You should get: $[0.875, 0.125]^T$, so 12.5% will be sick on Wednesday.

NOTE: If we were going to compute more powers of A, we might diagonalize it, but since we only needed A^2, we didn't need to do that in this problem.

14. If each row of A sums to the same number s, what is one eigenvalue and eigenvector? If each column of A sums to the same number s, does your previous answer hold?

SOLUTION: If each row sums to s, then $\mathbf{u} = [1, 1, \cdots, 1]^T$ is an eigenvector, since $A\mathbf{u} = s[1, 1, \cdots, 1]^T$. If each column sums to one, s is still an eigenvalue, since the eigenvalues of the matrix are the same as its transpose (and its transpose has rows that all sum to s). As we saw in class, the eigenvectors may change though.

15. If A is similar to B, show that they have the same eigenvalues.

SOLUTION: By definition, if A is similar to B, there is a matrix P such that

$$A = PBP^{-1}$$

If λ is an eigenvalue of A, then

$$A\mathbf{x} = \lambda\mathbf{x} \quad \Rightarrow \quad PBP^{-1}\mathbf{x} = \lambda\mathbf{x} \quad \Rightarrow \quad B(P^{-1}\mathbf{x}) = \lambda(P^{-1}\mathbf{x}) \quad \Rightarrow \quad B\mathbf{u} = \lambda\mathbf{u}$$

Therefore, λ is also an eigenvalue of B, but with a different eigenvector.

16. Prove that if the set $\{\mathbf{v}_1, \cdots \mathbf{v}_k\}$ form a basis for subspace W, and $\mathbf{x}$ is orthogonal to each $\mathbf{v}_i$, for $i = 1$ to k, then $\mathbf{x}$ is orthogonal to W. (Hint: Start with a generic vector $\mathbf{w} \in W$, and show that $\mathbf{x}\cdot\mathbf{w} = 0$.)

SOLUTION: Let $\mathbf{w} \in W$. Then we can write $\mathbf{w}$ in terms of the basis vectors,

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

Now take the dot product of both sides with $\mathbf{x}$ and simplify the right hand side:

$$\mathbf{w} \cdot \mathbf{x} = (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) \cdot \mathbf{x} = c_1\mathbf{v}_1 \cdot \mathbf{x} + c_2\mathbf{v}_2 \cdot \mathbf{x} + \cdots + c_k\mathbf{v}_k \cdot \mathbf{x} = 0 + 0 + \cdots + 0$$

Therefore, $\mathbf{x}$ is orthogonal to every vector in W.

17. Prove that the eigenvalues of a triangular matrix are the entries on its main diagonal.

    SOLUTION: By the properties of the determinant, we know that the determinant of a triangular matrix is the product of the diagonal entries. By taking the determinant of $A - \lambda I$, we only change the diagonal entries- That is, if A is triangular, then so is $A - \lambda I$. Therefore, the characteristic polynomial is given by:

$$(a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) = 0$$

    From which we get that $\lambda_1 = a_{11}, \lambda_2 = a_{22}$, and so on until $\lambda_n = a_{nn}$.

18. (More) True or False? If the statement is false and you can provide a counterexample to demonstrate this, then do so. If the statement is false and be can slightly modified so as to make it true then indicate how this may be done.

    (a) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent eigenvectors, then they correspond to distinct eigenvalues.

    SOLUTION: False. For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

    has two eigenvalues, $\lambda = 1, 1$, but two linearly independent eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$.

    If we had said "distinct eigenvalues" correspond to "linearly independent eigenvectors", the answer would have been "true".

    (b) If A is invertible, then A is diagonalizable.

    SOLUTION: False. Saying that A is invertible simply means that $\lambda = 0$ is not an eigenvalue of A. There is no direct connection between a matrix being invertible and being diagonalizable. For example, we can imagine that a 2×2 matrix has two eigenvalues, 1 and 0 (not invertible), but since we have distinct eigenvalues, the matrix is diagonalizable. On the other hand, we might have had complex eigenvalues (invertible), but the matrix is not diagonalizable (in the sense of D being a diagonal matrix in PDP^{-1}).

    (c) The orthogonal projection of $\mathbf{y}$ onto a vector $\mathbf{v}$ is the same as the orthogonal projection of $\mathbf{y}$ onto $c\mathbf{v}$ whenever $c \neq 0$.

    SOLUTION: True:

$$\text{Proj}_v(\mathbf{y}) = \frac{\mathbf{y}^T\mathbf{v}}{\mathbf{v}^T\mathbf{v}}\mathbf{v}$$

    And

$$\text{Proj}_{cv}(\mathbf{y}) = \frac{\mathbf{y}^T c\mathbf{v}}{c\mathbf{v}^T c\mathbf{v}} c\mathbf{v} = \frac{c^2}{c^2}\,\frac{\mathbf{y}^T\mathbf{v}}{\mathbf{v}^T\mathbf{v}}\mathbf{v} = \text{Proj}_v(\mathbf{y})$$

    (d) If A is an orthogonal matrix, then A^T is an orthogonal matrix.

    SOLUTION: True. Since A is orthogonal, then $A^T = A^{-1}$, so

$$I = AA^{-1} = AA^T = (A^T)^T A^T$$

    And therefore, A^T is orthogonal.

    (e) If A, B have the same eigenvalues, then they are similar.

    SOLUTION: If we had said "Similar implies the same eigenvalues", that would have been true. In this case, it depends- We might wonder if it is ever true? If both A, B are diagonalizable, I think we can show that the statement is true: In that case, we could write:

$$A = P_a D P_a^{-1} \qquad B = P_b D P_b^{-1}$$

where each D is the same, but the matrices P_a, P_b are possibly different. Solving each equation for D, we can write:

$$P_a^{-1} A P_a = P_b^{-1} B P_b \qquad \Rightarrow \qquad A = \left(P_a P_b^{-1}\right) B \left(P_b P_a^{-1}\right)$$

So in this particular case, it is true. In general, it wouldn't be since matrices A, B may not be diagonalizable (either or both may be defective)- For example,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

In this case, A, B have the same eigenvalues, A is diagonalizable, but B is not and A is not similar to B. If it were, then

$$A = PBP^{-1} \Rightarrow AP = PB \Rightarrow P = PB \Rightarrow I = B$$

but B is not the identity matrix.

NOTE: You wouldn't have to come up with the whole argument during an exam; this problem was just intended to get you to think about the relationship between similar matrices and eigenvalues.