HW Solutions, 1.8/9

1.8, 24 Suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. If T maps each $\mathbf{v}_i$ to $\mathbf{0}$, show that T maps everything to zero.

SOLUTION: Let $\mathbf{x}$ be an arbitrary vector in $\mathbb{R}^n$. We show that $T(\mathbf{x}) = \mathbf{0}$.

Since the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span $\mathbb{R}^n$, and $\mathbf{x} \in \mathbb{R}^n$, there exist constants $c_1, \ldots, c_p$ so that

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

By the linearity of T,

$$T(\mathbf{x}) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p) = 0 + 0 + \cdots + 0 = 0$$

1.8, 34 Show that if $\mathbf{u}, \mathbf{v}$ are linearly independent in $\mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation such that $T(\mathbf{u}), T(\mathbf{v})$ are linearly dependent, then $T(\mathbf{x}) = \mathbf{0}$ has a nontrivial solution.

Since $T(\mathbf{u}), T(\mathbf{v})$ are linearly dependent, we have constants c_1, c_2, not both zero, such that

$$c_1T(\mathbf{u}) + c_2T(\mathbf{v}) = 0$$

By the linearity of T, this means that $T(c_1\mathbf{u}) + c_2\mathbf{v}) = 0$.

Since $\mathbf{u}, \mathbf{v}$ are linearly independent and the constants c_1, c_2 are not both zero, $c_1\mathbf{u} + c_2\mathbf{v}$ cannot be zero. Therefore, T maps something that is not zero to zero.

1.9, 36 Show that the composition of linear maps is a linear map. That is, if $S : \mathbb{R}^p \to \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ are both linear, then $T(S(\mathbf{x}))$ is linear.

Proof: We show that the two parts of the definition are true.

$$T(S(\mathbf{x} + \mathbf{y})) = T(S(\mathbf{x}) + S(\mathbf{y})) \text{ by the linearity of } S$$

$$= T(S(\mathbf{x})) + T(S(\mathbf{y})) \text{ by the linearity of } T$$

Similarly,

$$T(S(c\mathbf{x})) = T(cS(\mathbf{x})) = cT(S(\mathbf{x}))$$

where the first equality holds because S is linear, the second holds because T is linear.