HW Solutions, 2.2

2.2, 16 Suppose that A, B are $n \times n$, B and AB are invertible. Show that A is invertible.

SOLUTION: Using the hint, if $C = AB$, then C is invertible, and

$$CB^{-1} = ABB^{-1} = A$$

Therefore, A is the product of the invertible matrix C and B^{-1}, so A is invertible.

2.2, 18 Suppose P is invertible and $A = PBP^{-1}$. Solve for B in terms of A.

SOLUTION: Be sure you multiply on the correct side:

$$A = PBP^{-1} \Rightarrow P^{-1}A = P^{-1}PBP^{-1} = BP^{-1} \Rightarrow P^{-1}AP = BP^{-1}P = B$$

2.2, 20 Suppose that A, B, X are $n \times n$ with A, X, and $A - AX$ invertible. Suppose that

$$(A - AX)^{-1} = X^{-1}B$$

(a) Show that B is invertible:
SOLUTION: Multiply on the left by X, and $X(A - AX)^{-1} = B$. Therefore, B is the product of invertible matrices (and is itself invertible).

(b) Solve the equation given above for X. If you need to invert a matrix, explain why.
SOLUTION: I don't recall what we did in class (there are multiple ways of expressing the solution), but here is one possibility. Multiply both sides by $(A - AX)$:

$$X(A - AX)^{-1} = XX^{-1}B \quad \Rightarrow \quad X(A - AX)^{-1} = B \quad \Rightarrow$$

$$X = B(A - AX) = BA - BAX \quad \Rightarrow \quad X + BAX = BA \quad \Rightarrow \quad (I + BA)X = BA$$

Is $I + BA$ invertible? The last equality implies that $I + BA = BAX^{-1}$, which is the product of three invertible matrices. Therefore

$$X = BA(I + BA)^{-1}$$

Alternative Solution: Another solution might be the following- First, invert both sides of the given equation, since we know that B is invertible:

$$A - AX = B^{-1}X \quad \Rightarrow \quad A = B^{-1}X + AX = (B^{-1} + A)X$$

This last equation tells us that $B^{-1} + A = AX^{-1}$, so it is also invertible. Now do the inversion:

$$X = (B^{-1} + A)^{-1}A$$

Just for fun: Can you make the two solutions look alike?