Some Proofs from Chapter 2

2.14 A proof by induction proceeds logically by (i) Prove the statement true for some base case (typically 1 or 2), then (ii)-(iii): Assume that if the statement is true for case n, then it must be true for $n+1$. Notice the logic here- Once proven for $n=2$, if we prove that the transition from n to $n+1$ is always true, then the statement is true for $n=3$. If it is true for $n=3$, then it must also be true for $n=4$, and so on.

The statement we wish to prove by induction is:

$$P(E_1 \cup E_2 \cup \ldots \cup E_n) \le P(E_1) + P(E_2) + \ldots + P(E_n)$$

First we'll prove it for a pair of sets (it is trivially true for only one set):

$$\begin{aligned} P(E_1 \cup E_2) &= P(E_1) + P(E_2) - P(E_1 \cap E_2) \\ &\le P(E_1) + P(E_2) \end{aligned}$$

We might note that we get equality only if the sets are mutually exclusive (m.e.)

Next we prove the transition: Assuming the statement is true for n sets, it must be true for $n+1$ sets. That means that, for the transition, we assume that:

$$P(E_1 \cup E_2 \cup \ldots \cup E_n) \le P(E_1) + P(E_2) + \ldots + P(E_n)$$

And from this, we must prove that:

$$P(E_1 \cup E_2 \cup \ldots \cup E_n \cup E_{n+1}) \le P(E_1) + P(E_2) + \ldots + P(E_n) + P(E_{n+1})$$

So we start with the left-hand side of the expression. We will then group the sets in a suggestive manner:

$$P(E_1 \cup E_2 \cup \ldots \cup E_n \cup E_{n+1}) = P\left((E_1 \cup E_2 \cup \ldots \cup E_n) \cup E_{n+1}\right) =$$

$$P\left(E_1 \cup E_2 \cup \ldots \cup E_n\right) + P(E_{n+1}) - P\left((E_1 \cup E_2 \cup \ldots \cup E_n) \cap E_{n+1}\right) \le$$

$$P\left(E_1 \cup E_2 \cup \ldots \cup E_n\right) + P(E_{n+1}) \le$$

$$P(E_1) + P(E_2) + \ldots + P(E_n) + P(E_{n+1})$$

2.22 Show that if A, B are independent then

- A' and B are independent.
  We need to show that $P(A' \cap B) = P(A')P(B)$. Working backwards, let's see if we can find the right relationship:

$$\begin{aligned} P(A' \cap B) &= P(A')P(B) && \text{What we want} \\ &= (1 - P(A))P(B) && \text{Convert } A' \text{ to } A \\ &= P(B) - P(A)P(B) && \\ &= P(B) - P(A \cap B) && \text{Independence} \end{aligned}$$

There it is- Here is the correct direction now:

$$\begin{aligned} P(A' \cap B) + P(A \cap B) &= P(B) && \text{M.E. sets} \\ P(A' \cap B) &= P(B) - P(A \cap B) \\ &= P(B) - P(A)P(B) && \text{Indep of } A, B \\ &= (1 - P(A))P(B) \\ &= P(A')P(B) \end{aligned}$$

Therefore, A' and B are independent.

- A' and B' are independent. In this case, show that

$$P(A' \cap B') = P(A')P(B')$$

We might be able to do this straight off:

$$\begin{aligned} P(A' \cap B') &= P((A \cup B)') && \text{M.E. sets} \\ &= 1 - P(A \cup B) \\ &= 1 - (P(A) + P(B) - P(A \cap B) \\ &= 1 - P(A) - P(B) + P(A)P(B) && \text{Indep of } A, B \\ &= (1 - P(A)) - P(B)(1 - P(A)) \\ &= P(A')P(B') \end{aligned}$$

2.27 If A, B, C are independent, show that:

(a) A and $B \cap C$ are also independent.

SOLUTION: First you might write down what it is we need to show. In this case, by the definition of independence, we need:

$$P(A \cap (B \cap C)) = P(A)P(B \cap C)$$

We notice that, from independence of all sets, $P(B \cap C) = P(B)P(C)$. I think we can prove this:

$$\begin{aligned} P(A \cap (B \cap C)) &= P(A \cap B \cap C) && \text{Standard set theory} \\ &= P(A)P(B)P(C) && \text{3-way independence} \\ &= P(A)P(B \cap C) && \text{2-way indep from the full indep} \end{aligned}$$

Conclusion: We have shown that A and $B \cap C$ are independent, if A, B, C are independent.

(b) A and $B \cup C$ are independent.

It is easiest to prove this by using two theorems-

- If A, B are independent, so are A' and B' (Proved in class).
- Look at part (a).

We are done if we can show that A' and $(B \cup C)'$ are independent, from our first theorem.

We see that: $(B \cup C)' = B' \cap C'$. Notice that, by part (a), that if A', B' and C' are independent, then so are A' and $B' \cap C'$.

2.32 Prove Theorem 2.12: If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S and $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event A in S we can write:

$$P(A) = \sum_{i=1}^{k} P(B_i)P(A|B_i)$$

(Note the word *partition*, defined in a footnote on page 9, means that the B_i's are mutually exclusive and the union is the entire sample space. Try to draw a Venn diagram of the situation).

For any event A, since the sets B_i form a partition, we can write A as a union of mutually exclusive sets:

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \ldots (A \cap B_k)$$

so that the probability becomes a sum:

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \ldots P(A \cap B_k)$$

Each of these can be written as a conditional probability as long as the probability of each B_i is not zero. We check what we are trying to prove, and we notice that:

$$P(A \cap B_i) = P(B_i)P(A|B_i)$$

so that the statement is proven by rewriting our previous sum:

$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + \ldots P(B_k)P(A|B_k)$$

1. **Extra Practice:** Prove that, if A and B are independent, then so are A and B'. (Try to work it out before you look at the solution below)

SOLUTION: We are given that A, B are independent. This means that we are given:

$$P(A \cap B) = P(A)P(B)$$

We need to show that this implies:

$$P(A \cap B') = P(A)P(B')$$

We might work backwards a bit to see where it leads us:

$$P(A \cap B') = P(A)P(B') = P(A)(1 - P(B)) = P(A) - P(A)P(B)$$

Hmmm... This might be helpful. Notice that our last equality implies that:

$$P(A) = P(A)P(B) + P(A \cap B')$$

but by independence, we can substitute $P(A \cap B)$ for $P(A)P(B)$:

$$P(A) = P(A)P(B) + P(A \cap B') = P(A \cap B) + P(A \cap B')$$

This would be true as long as we can write

$$A = (A \cap B) \cup (A \cap B')$$

and $(A \cap B)$, $(A \cap B')$ are mutually exclusive (which are both true- Look at the Venn Diagram, or Exercise 2.4, p. 30). There is our proof- Now we'll go in the right direction (backwards):

We know that $A \cap B$ and $A \cap B'$ are mutually exclusive sets, and

$$A = (A \cap B) \cup (A \cap B')$$

Therefore,

$$P(A) = P(A \cap B) + P(A \cap B')$$

By the independence of A, B:

$$P(A) = P(A \cap B) + P(A \cap B') = P(A)P(B) + P(A \cap B')$$

Work through the algebra:

$$P(A \cap B') = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(B')$$

Therefore, A and B' are also independent.

2. (Additional Exercise:) Show by means of numerical examples that $P(B|A) + P(B|A')$ may or may not be equal to one. Additionally, try to prove that: $P(A|B)+P(A'|B) = 1$

SOLUTION to the proof: Notice that by working backwards slightly,

$$P(A|B) + P(A'|B) = \frac{P(A \cap B)}{P(B)} + \frac{P(A' \cap B)}{P(B)} = \frac{P(A \cap B) + P(A' \cap B)}{P(B)}$$

Do you see that the numerator is actually $P(B)$? Here is the direct proof:

We can write B as the union of mutually exclusive sets:

$$B = (B \cap A) \cup (B \cap A') = (A \cap B) \cup (A' \cap B)$$

Therefore,

$$1 = \frac{P(B)}{P(B)} = \frac{P(A \cap B) + P(A' \cap B)}{P(B)} = P(A|B) + P(A'|B)$$