Misc Exercises from Chapter 4

4.21 Prove $\mathrm{var}(aX+b) = a^2\sigma^2$

To keep our rvs straight, let $Y = aX + b$. Then we want to show that

$$\sigma_Y^2 = a^2\sigma_X^2$$

First, compute the mean, μ_Y:

$$\mu_Y = E(aX+b) = aE(X) + b = a\mu_X + b$$

Now,

$$Y - \mu_y = aX + b - (a\mu_X + b) = a(X - \mu_X)$$

Therefore,

$$E((Y-\mu_Y)^2) = E((a(X-\mu_X))^2) = a^2E((X-\mu_X)^2)$$

which is another way to say that:

$$\sigma_Y^2 = a^2\sigma_X^2$$

4.24 Let $f(x) = 2x^{-3}$ for $x > 1$ (zero elsewhere). Compute the mean and variance, if they exist:

$$\mu = \int_{-\infty}^{\infty} xf(x)\,dx = 2\int_1^{\infty} x^{-2}\,dx = -\frac{2}{x}\Big|_1^{\infty} = 0 - (-2) = 2.$$

As in class, rather than computing σ^2 directly, compute the second moment about the origin, $E(X^2)$, then use the theorem:

$$\sigma^2 = E(X^2) - \mu^2$$

So,

$$E(X^2) = 2\int_1^{\infty} \frac{1}{x}\,dx = 2\ln(x)$$

and the limit does not exist as $x \to \infty$. Therefore, the variance will not exist (but you probably anticipated that, since the original function was $2/x^3$?)

4.25 Show that formula in the book holds. HINT: Try doing the second part first- Find formulas for μ_3 and μ_4 before looking for the general case:

$$\mu_3 = E((X-\mu)^3)$$

Expand this using the binomial theorem:

$$E(X^3 - 3X^2\mu + 3X\mu^2 - \mu^3) = E(X^3) - 3E(X^2)\mu + 3E(X)\mu^2 - \mu^3$$

The straight binomial pattern is changed because $E(X) = \mu$, which simplifies to:

$$\mu_3 = \mu_3' - 3\mu_2'\mu + 3\mu^3 - \mu^3 = \mu_3' - 3\mu_2'\mu + 2\mu^3$$

Similarly, for μ_4', expand $E((X-\mu)^4)$ (the line from Pascal's triangle is $1, 4, 6, 4, 1$):

$$E(X^4 - 4X^3\mu + 6X^2\mu^2 - 4X\mu^3 + \mu^4) =$$

$$\mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 4\mu^4 + \mu^4 = \mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4$$

And the general pattern is to expand $(X-\mu)^r$ out using the binomial theorem, then use the linearity of E, then combine the last two terms in the sum:

$$X^r + \binom{r}{1}X^{r-1}(-\mu) + \binom{r}{2}X^{r-2}(-\mu)^2 +$$

$$+\cdots+\binom{r}{r-1}X(-\mu)^{r-1} + \binom{r}{r}(-\mu)^r$$

The $(-\mu)^i$ term gives us the $(-1)^i$ term from the formula in the text, and taking the expected value of the sum of the last two terms should finish us up:

$$E\left(\binom{r}{r-1}X(-\mu)^{r-1} + \binom{r}{r}(-\mu)^r\right) = (-1)^{r-1}r\mu^r + (-1)^r\mu^r$$

Factor to get: $(-1)^{r-1}\mu^r\,(r-1)$, which is the last term of the formula in the text.

4.29 Prove Markov's Inequality: Let X be a nonnegative rv. Then for every given number $k > 0$,

$$P(X \geq k) \leq \frac{E(X)}{k}$$

We will prove this in the case of a discrete distribution, and you'll see how it works. Given $k > 0$, we can write:

$$E(X) = \sum_x f(x) = \sum_{x<k} xf(x) + \sum_{x\geq k} xf(x) \geq \sum_{x\geq k} xf(x)$$

Note that to get that last inequality, we had to assume that X has only nonnegative values!

Now, by replacing each of those x's in the last sum by the constant k (notice that k is smaller than all the x's), we get a sum that is smaller,

$$E(X) \geq \sum_{x\geq k} xf(x) \geq \sum_{x\geq k} kf(x) = kP(X \geq k)$$

4.34 Same issue as 4.33 (find the mgf), but with: $f(x) = 1$ for $0 < x < 1$, zero elsewhere.

$$M(t) = E(\mathrm{e}^{tX}) = \int_0^1 \mathrm{e}^{tx}\,dx = \left.\frac{1}{t}\mathrm{e}^{tx}\right|_0^1 = \frac{\mathrm{e}^t - 1}{t}$$

We notice that the first two derivatives are:

$$M'(t) = \frac{\mathrm{e}^t(t-1)+1}{t^2} \qquad M''(t) = \frac{\mathrm{e}^t(t^2-2t+2)-2}{t^3}$$

The derivatives do not exist at $t = 0$, which gives us a bit of a mystery. There are two alternatives for dealing with it:

- Use l'Hospital's rule, to take the limit as $t \to 0$:

$$\lim_{t\to 0} M'(t) = \lim_{t\to 0} \frac{t\mathrm{e}^t + \mathrm{e}^t - \mathrm{e}^t}{2t} = \frac{1}{2}$$

$$\lim_{t\to 0} M''(t) = \lim_{t\to 0} \frac{(2t-2)\mathrm{e}^t + \mathrm{e}^t(t^2-2t+2)}{3t^2} = \lim_{t\to 0} \frac{\mathrm{e}^t}{3} = \frac{1}{3}$$

- Use a Maclaurin series of e^t and pull off the necessary coefficients. Begin with the exponential,

$$\mathrm{e}^t = 1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3 + \cdots$$

Subtract 1, then divide by t:

$$\mathrm{e}^t - 1 = t + \frac{1}{2}t^2 + \frac{1}{3!}t^3 + \cdots$$

$$\frac{\mathrm{e}^t - 1}{t} = 1 + \frac{1}{2}t + \frac{1}{3!}t^2 + \cdots$$

Think of this as if it were the Maclaurin series for a given function. Then

$$M(0) = 1 \qquad M'(0) = \frac{1}{2} \qquad \frac{M''(0)}{2} = \frac{1}{3 \cdot 2}$$

so that $\mu_1' = \frac{1}{2}$ and $\mu_2' = \frac{1}{3}$, just as before.

Now we can easily compute the variance,

$$\sigma^2 = \mu_2' - \mu^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

4.37 Given the pdf $f(x) = \frac{1}{2}\mathrm{e}^{-|x|}$, find its mgf (form given in the problem):

$$M(t) = E(\mathrm{e}^{tX}) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{tx}\mathrm{e}^{-|x|}\,dx =$$

Split the integral:

$$\frac{1}{2}\int_{-\infty}^{0} \mathrm{e}^{tx}\mathrm{e}^{x}\,dx + \frac{1}{2}\int_{0}^{\infty} \mathrm{e}^{tx}\mathrm{e}^{-x}\,dx = \frac{1}{2}\int_{-\infty}^{0} \mathrm{e}^{(t+1)x}\,dx + \frac{1}{2}\int_{0}^{\infty} \mathrm{e}^{(t-1)x}\,dx =$$

Taking these limits bit by bit (this is where we need to be careful):

$$\frac{1}{2}\left(\frac{1}{t+1}\mathrm{e}^{(t+1)x}\bigg|_{-\infty}^{0}\right) = \frac{1}{2}\left(\frac{1}{t+1} - \lim_{A\to-\infty}\frac{\mathrm{e}^{(t+1)A}}{t+1}\right)$$

This limit exists when $t+1 > 0$ (since A is negative). Therefore, if $t > -1$, then this part of the integral becomes $1/(2(t+1))$.

Similarly, the positive x−axis gives:

$$\frac{1}{2}\left(\frac{1}{t-1}\mathrm{e}^{(t-1)x}\bigg|_{0}^{\infty}\right) = \frac{1}{2}\left(\lim_{A\to\infty}\frac{\mathrm{e}^{(t-1)A}}{t-1} - \frac{1}{t-1}\right)$$

This limit will exist when $t - 1 < 0$, or when $t < 1$. In that case, this part of the integral gives $-1/(2(t-1))$.

Put it together to get the text's suggestion. If $-1 < t < 1$, then

$$M(t) = \frac{1}{2(t+1)} - \frac{1}{2(t-1)} = \frac{1}{1-t^2}$$

4.38 We want to find the variance (so also the mean) to the previous problem by using the mgf. In the first technique, get the Maclaurin series and read off the coefficients. In the second, use derivatives directly.

The series can be found by first taking the series for $1/(1-t)$. Notice that this is the sum of a geometric series:

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots = \sum_{n=0}^{\infty} t^n$$

substitute t^2 for t to get our series:

$$\frac{1}{1-t^2} = 1 + t^2 + t^4 + t^6 + \cdots = M(0) + M'(0)t + \frac{M''(0)}{2}t^2 + \cdots$$

(Note the typo in the solutions manual for the power series) so $M'(0) = \mu = 0$ and $mu_2' = E(X^2) = M''(0) = 2$. Since the mean is zero, this is also the variance.

To use the derivative "technique",

$$M(t) = \frac{1}{1-t^2} \qquad M'(t) = \frac{2t}{(1-t^2)^2} \qquad M''(t) = \frac{6t^2+2}{(1-t^2)^3}$$

from which we get the same numbers.

4.39 Prove Theorem 4.10, or we'll prove the more general case (Part 3). That is, let X be an rv, $Y = aX + b$. Then

$$M_Y(t) = E(\mathrm{e}^{Yt}) = E(\mathrm{e}^{(aX+b)t}) = E(\mathrm{e}^{X(at)}\mathrm{e}^{bt})$$

Now, e^{bt} is constant with respect to the expected value operator (either an integral or a sum in x), so it factors out, and we're done:

$$E(\mathrm{e}^{X(at)}\mathrm{e}^{bt}) = \mathrm{e}^{bt}E(\mathrm{e}^{X(at)}) = \mathrm{e}^{bt}M_X(at)$$

Notice that we already know what multiplying by a constant and a shift does to the mean and variance. Does this result reflect those changes? Let's compute the mean and variance of Y using the mgf by computing the derivatives (then evaluate at $t = 0$):

$$M_Y'(t) = b\mathrm{e}^{bt}M_X(at) + \mathrm{e}^{bt}M_X'(at) \cdot a$$

$$M_Y'(0) = bM_X(0) + aM_X'(0) = b \cdot 1 + a\mu_X = a\mu_X + b$$

Similarly, for the second derivative:

$$M_Y''(0) = b^2 + 2ab\mu_x + a^2\mu_2'$$

4.40 Use the previous formula to solve this problem. The mgf is:

$$M_X(t) = \mathrm{e}^{3t+8t^2}$$

and $Z = \frac{1}{4}(Z-3) = \frac{1}{4}Z - \frac{3}{4}$. Then:

$$M_Z(t) = \mathrm{e}^{-3/4\,t}M_X\left(\frac{t}{4}\right) = \mathrm{e}^{-\frac{3t}{4}}\mathrm{e}^{3\left(\frac{t}{4}\right)+8\left(\frac{t^2}{16}\right)} = \mathrm{e}^{\frac{1}{2}t^2}$$

The series expansion is:

$$1 + \left(\frac{1}{2}t^2\right) + \frac{1}{2}\left(\frac{1}{2}t^2\right)^3 + \cdots$$

Therefore, $\mu = 0$ and $\sigma^2 = 1$ (since $\mu = 0$).

4.44 Compute the covariance of the pdf in Exercise 3.74. That is, $f(x, y) = (1/4)(2x + y)$, for $0 < x < 1, 0 < y < 2$, zero elsewhere.

We might compute the marginal distributions first:

$$f_1(x) = \frac{1}{4}\int_0^2 2x + y\,dy = \frac{1}{4}\left(2xy + \frac{1}{2}y^2\right|_0^2 = \frac{1}{4}(4x + 2 - 0) = x + \frac{1}{2}$$

$$f_2(y) = \frac{1}{4}\int_0^1 2x + y\,dx = \frac{1}{4}\left(x^2 + xy\right|_0^1 = \frac{1}{4}(y + 1 - 0) = \frac{1}{4}(y + 1)$$

Now, respective means:

$$\mu_x = \int_0^1 x(x + 1/2)\,dx = \int_0^1 x^2 + \frac{1}{2}x\,dx = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

$$\mu_y = \frac{1}{4}\int_0^2 y(y + 1)\,dy = \frac{1}{4}\left(\frac{1}{3}y^3 + \frac{1}{2}y^2\right|_0^2 = \frac{7}{6}$$

We also need $E(XY)$:

$$E(XY) = \frac{1}{4}\int_0^1\int_0^2 xy(2x + y)\,dy\,dx = \int_0^1 \frac{2}{3}x + x^2\,dx = \frac{2}{3}$$

Therefore,

$$\sigma_{xy} = E(XY) - \mu_x\mu_y = \frac{2}{3} - \frac{7}{12}\cdot\frac{7}{6} = -\frac{1}{72}$$

4.47 The joint mgf for n rvs $X_1, \ldots, X_n$ is

$$E(\exp(t_1X_1 + \ldots + t_nX_n))$$

$$\frac{\partial}{\partial t_j}\left(\sum_{x_1,\ldots,x_n} \exp(t_1x_1 + \ldots + t_nx_n)\,f(x_1, \ldots, x_n)\right) =$$

$$\sum_{x_1,\ldots,x_n} x_j\mathrm{e}^{(t_1x_1+\ldots+t_nx_n)}f(x_1, \ldots, x_n)\Bigg|_{t_1=0,\ldots,t_n=0} = \sum_{x_1,\ldots,x_n} x_jf(x_1, \ldots, x_n)$$

And this is $E(X_j)$. Similarly, the second partial derivative will give an x_ix_j in front of the pdf in the sum (or integral).

The joint pdf e^{-x-y} for $x > 0, y > 0$, and zero elsewhere. The mgf is :

$$\int_0^\infty\int_0^\infty \mathrm{e}^{t_1x+t_2y}\mathrm{e}^{-x-y}\,dx\,dy = \int_0^\infty \mathrm{e}^{-(1-t_1)x}\,dx\int_0^\infty \mathrm{e}^{-(1-t_2)y}\,dy =$$

$$\int_0^\infty \mathrm{e}^{-(1-t_1)x}\,dx = \frac{1}{1 - t_1}\,\mathrm{e}^{-(1-t_1)x}\Big|_0^\infty = \frac{-1}{1 - t_1}$$

Convergence: Depends on $t_1 < 1$.

Overall, the mgf is:

$$M_{xy}(t_1, t_2) = \frac{1}{(1 - t_1)(1 - t_2)} \qquad \frac{\partial M}{\partial t_1} = \frac{1}{(1 - t_1)^2(1 - t_2)}$$

Now compute the moments:

(a) $\mu_x = E(X) = \dfrac{\partial M}{\partial t_1}(0,0) = \left.\dfrac{1}{(1-t_1)^2(1-t_2)}\right|_{0,0} = 1$

(b) $\mu_y = E(Y) = \dfrac{\partial M}{\partial t_2}(0,0) = \left.\dfrac{1}{(1-t_1)(1-t_2)^2}\right|_{0,0} = 1$

(c) $E(XY) = \dfrac{\partial^2 M}{\partial t_1 \partial t_2}(0,0) = \left.\dfrac{1}{(1-t_1)^2(1-t_2)^2}\right|_{0,0} = 1$

(d) $\sigma_{xy} = E(XY) - \mu_x\mu_y = 1 - 1 = 0$ (Or, observe that X and Y are independent, but we wanted to do out the computations this time).

4.48 If X_1, X_2, X_3 are independent, and have means and vars given in the table, find the mean and variance of Y and Z below.

	X_1	X_2	X_3
μ	4	9	3
σ^2	3	7	5

- $Y = 2X_1 - 3X_2 + 4X_3$

$$\mu_y = E(Y) = 2\mu_{x_1} - 3\mu_{x_2} + 4\mu_{x_3} = 8 - 27 + 12 = -7$$

In this exercise, assume independence, so that $E((X_i - \mu_{x_i})(X_j - \mu_{x_j})) = 0$:

$$\sigma_y^2 = 2^2\sigma_{x_1}^2 + (-3)^2\sigma_{x_2}^2 + 4^2\sigma_{x_3}^2 = 4 \cdot 3 + 9 \cdot 7 + 16 \cdot 5 = 155$$

- $Z = X_1 + 2X_2 - X_3$

$$\mu_z = \mu_{x_1} + 2\mu_{x_2} - \mu_{x_3} = 4 + 18 - 3 = 19$$

$$\sigma_z^2 = 3 + 4 \cdot 7 + 5 = 36$$

4.49 Same as above, but drop the assumption of independence and let the covariances be:

$$\sigma_{12} = 1 \quad \sigma_{23} = -2 \quad \sigma_{13} = -3$$

Let $\hat{X}_j = X_j - \mu_{x_j}$. Then

$$\sigma_y^2 = E((2\hat{X}_1 - 3\hat{X}_2 + 4\hat{X}_3)^2)$$

Multiply this out algebraically (leaving off the hats):

$$\begin{array}{ccccccc}
2^2X_1^2 & - & 6X_1X_2 & + & 8X_1X_3 & + & \\
3^2X_2^2 & - & 6X_1X_2 & & & - & 12X_2X_3 \\
4^2X_3^2 & & & + & 8X_1X_3 & - & 12X_2X_3 \\
\hline
155 & - & 12 \cdot 1 & + & 16 \cdot (-3) & - & 24 \cdot (-2) = 143
\end{array}$$

Note that we see $2a_ia_jX_iX_j$ for each mixed term. If

$$Z = X_1 + 2X_2 - X_3$$

then we will have:

$$36 + 2 \cdot 2 \cdot 1 + 2 \cdot (1)(-1)(-3) + 2 \cdot (2)(-1)(-2) = 54$$

4.51 Prove Theorem 4.15: Given $X_1, \dots, X_n$ rvs, and Y_1, Y_2 are two linear combinations of the rvs, give the covariance between Y_1 and Y_2.

Assume that Y_1 and Y_2 have been mean subtracted (this implies that each of the n rvs have been mean subtracted). Then

$$\sigma_{y_1 y_2} = E(Y_1 Y_2) = E((a_1 X_1 + \dots a_n X_n)(b_1 X_1 + \dots b_n X_n))$$

Every term of this expansion will be of the form:

$$c_{ij} X_i X_j = \begin{cases} a_j b_j X_j^2 & \text{if } j = i \\ (a_i b_j + a_j b_i) X_i X_j & \text{if } i \neq j \end{cases}$$

$$= \begin{cases} a_j b_j \sigma_{x_j}^2 & \text{if } j = i \\ (a_i b_j + a_j b_i) \sigma_{x_i, x_j} & \text{if } i \neq j \end{cases}$$

Sum all the terms to get the result.

4.53 Given three rvs such that:

$$\begin{array}{lll} \sigma_{x_1}^2 = 5 & \sigma_{x_1 x_2} = 3 & x_2, x_3 \text{ indep} \\ \sigma_{x_2}^2 = 4 & \sigma_{x_1 x_3} = -2 & \\ \sigma_{x_3}^2 = 7 & & \end{array}$$

Find the covariance of Y_1, Y_2 given in the top, bottom lines below resp.:

$$\begin{array}{ccccc} & X_1 - & 2X_2 + & 3X_3 & \\ & -2X_1 + & 3X_2 + & 4X_3 & \\ \hline (-2)5+ & (4)(3)- & 6(-2) & & \\ (-6)4+ & (3)(3)+ & 9(0) & & \\ (12)7 & + & 4(-2)+ & (-8)(0) & \end{array}$$

Answer: 75

4.57 For the pdf from 3.74, $f(x, y) = \frac{1}{4}(2x + y)$, $0 < x < 1$, $0 < y < 2$, and zero elsewhere, find the conditional mean and variance of y, given $x = 1/4$.

First, we find the conditional pdf, evaluate at $x = 1/4$, then simply find the usual mean and variance:

$$f(y|x) = \frac{f(x, y)}{f_1(x)}$$

and the marginal pdf, $f_1(x)$ is:

$$\int_0^2 \frac{1}{4}(2x + y)\, dy = x + \frac{1}{2}$$

Now,

$$f(y|x = 1/4) = \frac{\frac{1}{4}(y + \frac{1}{2})}{3/4} = \frac{1}{3}(y + \frac{1}{2})$$

Now the mean:

$$\frac{1}{3} \int_0^2 y^2 + \frac{1}{2} y\, dy = \frac{11}{9}$$

and the variance from the moment about the origin:

$$E(Y^2) \frac{1}{3} \int_0^2 y^3 + \frac{1}{2} y^2\, dy = \frac{16}{9}$$

Therefore,

$$\sigma_{y|x=1/4}^2 = \frac{16}{9} - \frac{11^2}{9^2} = \frac{23}{81}$$

4.61 We're computing the expected payoff. We note the uniform probability for the rolls of the dice, so we'll get \$10.00 one third of the time, and pay p dollars on the 1, 2, 5 or 6 (or 2/3 of the time):

$$10 \cdot \frac{1}{3} + p\frac{2}{3} = \frac{10 + 2p}{3}$$

To make the game equitable must mean that our expected payoff is zero, so p must be -\$5.00

4.63 Contractor's profit is $f(x) = (x+1)/18, [-1 < x < 5]$, zero elsewhere. I suppose we're assuming x is something like number of days? We want the expected value:

$$\int_{-1}^{5} x f(x)\, dx = \frac{1}{18}\int_{-1}^{5} x^2 + x\, dx = 3,000$$

4.67 Adams and Smith are betting on repeated flips of a coin. At the start, Adams has a dollars, Smith has b. At each flip, the loser pays the dollar and the game continues until either player is "ruined". Making use of the fact that an equitable game has an expectation of zero, find the probability that Adams wins before he loses all of his money.

Let p be the probability that Adams wins Smith's b dollars before he loses all his a dollars. Then the expected value would be given by:

$$pb + (1-p)(-a) = 0$$

from which we see that $p = \frac{a}{a+b}$

4.69 For practice, we'll do both the standard technique using the definitions of mean and variance, as well as mgf technique, where $f = (1/4)\mathrm{e}^{-x/4}$ for $x > 0$.

$$E(X) = \frac{1}{4}\int_{0}^{\infty} x\mathrm{e}^{-\frac{x}{4}}\, dx \qquad \begin{array}{c|c|c} + & x & (1/4)\mathrm{e}^{-x/4} \\ - & 1 & -\mathrm{e}^{-x/4} \\ + & 0 & 4\mathrm{e}^{-x/4} \end{array}$$

Therefore, we have:

$$-(x+4)\mathrm{e}^{-x/4}\Big|_0^{\infty} = 0 - -4 = 4$$

Similarly,

$$E(X^2) = \frac{1}{4}\int_{0}^{\infty} x^2\mathrm{e}^{-\frac{x}{4}}\, dx \qquad \begin{array}{c|c|c} + & x^2 & (1/4)\mathrm{e}^{-x/4} \\ - & 2x & -\mathrm{e}^{-x/4} \\ + & 2 & 4\mathrm{e}^{-x/4} \\ - & 0 & -16\mathrm{e}^{-x/4} \end{array}$$

Therefore, we have:

$$-(x^2 + 8x + 32)\mathrm{e}^{-x/4}\Big|_0^{\infty} = 32$$

So that the mean is 4 and the variance is $32 - 4^2 = 16$.

Using the mgf technique for this problem, we would have:

$$M_X(t) = \frac{1}{4}\int_{0}^{\infty} \mathrm{e}^{xt}\mathrm{e}^{-x/4}\, dx = \frac{1}{4}\int_{0}^{\infty} \mathrm{e}^{-x\left(\frac{1}{4}-t\right)}\, dx = \frac{1}{4}\cdot\frac{-1}{\frac{1}{4}-t}\mathrm{e}^{-x\left(\frac{1}{4}-t\right)}\Big|_0^{\infty}$$

We see that:

$$M_X(t) = \frac{1}{1-4t}$$

Using the geometric series analogy,

$$M_X(t) = 1 + (4t) + (4t)^2 + (4t)^3 + \cdots$$

Therefore, the mean is 4 and $\mu_2' = E(X^2) = 16 \cdot 2 = 32$, which again gives the variance as 16.

4.73 Given $\mu = 124$ with $\sigma = 7.5$, find the probability that X will be between 64 and 184.

First, convert to a form so we can apply Chebyshev:

$$64 < X < 184 \quad \Rightarrow \quad 124 - 60 < X < 124 + 60 \quad \Rightarrow \quad |X - \mu| < 60 = 8\sigma$$

The probability of this is $1 - \frac{1}{64} = \frac{63}{64}$.

4.76 **Hint: You might first set up pdfs for Z and W separately- Don't.** That is, when we compute the covariance, we need to have the same number of values for z and w. Best to write a small table to summarize your information:

	z	w	Prob
(T,T)	0	0	0.36
(T,H)	0	1	0.24
(H,T)	1	1	0.24
(H,H)	1	2	0.16

Now we can read off the arithmetic operations needed:

$$\mu_x = 0 \cdot 0.36 + 0 \cdot 0.24 + 1 \cdot 0.24 + 1 \cdot 0.16 = 0.4$$

$$\mu_y = 0 \cdot 0.36 + 1 \cdot 0.24 + 1 \cdot 0.24 + 2 \cdot 0.16 = 0.8$$

And the covariance (by way of $E(WZ)$):

$$E(WZ) = 0 \cdot 0 \cdot 0.36 + 0 \cdot 1 \cdot 0.24 + 1 \cdot 1 \cdot 0.24 + 1 \cdot 2 \cdot 0.16 = 0.56$$

Therefore,

$$\sigma_{wz} = 0.56 - 0.4 \cdot 0.8 = 0.24$$

4.77 Oops! Not assigned, but a nice easy problem...

The inside diameter of a tube is a rv with a mean of 3 inches and a std of 0.02 in. The thickness of the tube is a rv with a mean of 0.3 inches and a std of 0.0005 inches. The two rvs are independent.

Find the mean and std of the outside diameter of the tube.

SOLUTION: If X is the rv for the inside diameter, and Y is the rv for the thickness, then let Z be the rv for the outside diameter of the tube, where we see that (remember, diameters and not radii):

$$Z = X + 2Y$$

where $\mu_x = 3$, $\sigma_x = 0.02$, $\mu_y = 0.3$ and $\sigma_y = 0.0005$. Also, independence implies that the covariance is zero.

The mean and variance of Z:

$$\mu_z = \mu_x + 2\mu_y = 3.6 \text{ inches}$$

$$\sigma_z^2 = \sigma_x^2 + 4\sigma_y^2 = 0.000401$$

so that the standard deviation is approx. 0.02

4.78 Let L_i be the length of the ith brick, so that $\mu_{L_i} = 8$ and $\sigma_{L_i} = 0.1$. Let M_i be the width of the mortar between i and i+1 bricks, $\mu_{M_i} = 0.5$ and $\sigma_{M_i} = 0.003$.

Now,

$$W = \sum_{i=1}^{50} L_i + \sum_{k=1}^{49} M_k \quad \Rightarrow \quad \mu_W = 50\mu_L + 49\mu_M = 424.5 \text{ in} \approx 35.4 \text{ ft}$$

and

$$\sigma_W^2 = \sum_{k=1}^{50} \sigma_L^2 + \sum_{k=1}^{49} \sigma_M^2 \quad \Rightarrow \quad \sigma_W^2 = 50 \cdot (0.1)^2 + 49 \cdot (0.03)^2 = .5441$$

so that $\sigma_W \approx 0.7376$

Extra: Compare this result to that you get if

$$W = 50L + 49M$$

(By the way, why don't we use this as the model? This would imply that every brick has exactly the same size!)

No change in the mean, but a rather large change in the variance:

$$\sigma_W^2 = 50^2(0.1^2) + 49^2(0.03^2) = 27.169 \qquad \sigma_W = 5.211$$

Where did this come from? In the proof, notice what we said about $\text{cov}(X_i, X_j)$. If X_i and X_j are independent rvs, this quantity is zero (like for this problem). If X_i is the SAME as X_j, then the covariance is not zero, it is the variance.

4.79 If heads is a success when we flip a coin, getting a six is a success when we roll a die, and getting an ace from an ordinary deck of cards is success, then find the mean and standard deviation of the total number of successes when we:

(a) Flip, roll, and draw.

(b) Flip thrice, roll twice and draw once.

Most of this problem is in the construction of the pieces that are needed: Let X be the event of Heads, Y be 6 on the die, and Z be an ace.

In modelling each distribution, think of using 0 for failure and 1 for success, as we did in the examples below:

- In the coin flip, let $x = 0$ be failure (getting a Tail), and $x = 1$ be success (getting a Head). Then $P(x = 0) = \frac{1}{2}$ and $P(x = 1) = \frac{1}{2}$. Therefore,

$$\mu_X = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$$

$$E(X^2) = 0^2 \cdot \frac{1}{2} + 1^2 \cdot \frac{1}{2} = \frac{1}{2} \qquad \sigma_X^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

Similarly,

- In the roll of the die, let $y = 0$ be failure (getting anything except 6), and $x = 1$ be success (getting a 6). Then $P(x = 0) = \frac{5}{6}$ and $P(x = 1) = \frac{1}{6}$. Therefore,

$$\mu_Y = 0 \cdot \frac{5}{6} + 1 \cdot \frac{1}{6} = \frac{1}{6}$$

$$E(Y^2) = 0^2 \cdot \frac{5}{6} + 1^2 \cdot \frac{1}{6} = \frac{1}{6} \qquad \sigma_Y^2 = \frac{1}{6} - \frac{1}{36} = \frac{5}{36}$$

- In drawing an Ace, let $z = 0$ be failure (getting anything except A), and $x = 1$ be success (getting an A). Then $P(x = 0) = \frac{12}{13}$ and $P(x = 1) = \frac{1}{13}$. Therefore,

$$\mu_Z = 0 \cdot \frac{12}{13} + 1 \cdot \frac{1}{13} = \frac{1}{13}$$

$$E(Z^2) = 0^2 \cdot \frac{12}{13} + 1^2 \cdot \frac{1}{13} = \frac{1}{13} \qquad \sigma_Z^2 = \frac{1}{13} - \frac{1}{169} = \frac{12}{169}$$

Now we interpret parts A, B and substitute in the appropriate values:

(a) A flip, roll and draw: $W = X + Y + Z$. Noting that these events are independent,

$$\mu_W = \mu_X + \mu_Y + \mu_Z = \frac{1}{4} + \frac{1}{6} + \frac{1}{13} = \frac{58}{78} = 0.74$$

Independence means we can sum the variances:

$$\sigma_W^2 = \frac{1}{4} + \frac{5}{36} + \frac{12}{169} = 0.46 \qquad \sigma_W \approx 0.68$$

(b) In this case, $W = X_1 + X_2 + X_3 + Y_1 + Y_2 + Z$, where X_i is the coin toss, Y_i is the roll of the dice. Notice that this is different than saying

$$W = 3X + 2Y + Z$$

which implies that we each of the three X's are the same, etc. (Also see the note from Problem 4.78) This difference again does not appear in the mean, but has a significant impact on the standard deviation!

$$\mu_W = 3\mu_X + 2\mu_Y + \mu_Z = \frac{3}{4} + \frac{1}{3} + \frac{1}{13} = \frac{149}{78}$$

$$\sigma_W^2 = \sigma_x^2 + \sigma_X^2 + \sigma_X^2 + \sigma_Y^2 + \sigma_Y^2 + \sigma_Z^2$$

(We write it this way to emphasize the difference between this and:

$$\sigma_W^2 = 3^2\sigma_x^2 + 2^2\sigma_y^2 + \sigma_z^2$$

which would be the variance if $W = 3X + 2Y + Z$). Substitution gives 1.099 for the variance, 1.05 for the standard deviation.