Exercises: 8.1-8.2

8.5 Straightforward "Show this.." type problem. In this case, the results follow from the respective definitions.

8.7 (Hint was given in class to follow these steps)

- Show that the mean of each X_i is zero.

$$E(X_i) = \frac{1}{2}\left(1 - 2^{-i}\right) + \frac{1}{2}\left(2^{-i} - 1\right) = 0$$

- Show that the variance of each X_i can be written as:

$$1 - \left(\frac{1}{2}\right)^{i-1} + \left(\frac{1}{4}\right)^{i}$$

Compute it:

$$E(X_i^2) = \frac{1}{2}\left(1 - 2^{-i}\right)^2 + \frac{1}{2}\left(2^{-i} - 1\right)^2 = \left(1 - \frac{1}{2^i}\right)^2$$

and expand.

- Show that the variance of Y_n can be written as

$$n - 2 + \frac{1}{3} + A_n$$

where $A_n \to 0$ as $n \to \infty$.

$$E(Y_n^2) = \sum_{i=1}^{n} E(X_i^2) = \sum_{i=1}^{n}\left(1 - \left(\frac{1}{2}\right)^{i-1} + \left(\frac{1}{4}\right)^{i}\right) =$$

$$n - \frac{1-(1/2)^n}{1-\frac{1}{2}} + \frac{1}{4}\frac{1-(1/4)^n}{1-\frac{1}{4}} = n - 2 + \frac{1}{3} + \left(\frac{1}{2^{n-1}} - \frac{1}{3}\cdot\frac{1}{4^n}\right)$$

The term in parentheses is A_n, which goes to zero as $n \to \infty$. The rest goes to ∞.

Class Note: So what does this show? There were sufficient conditions (independent, uniformly bounded, variance of the sum becomes infinite), then the distribution of the standardized mean approaches standard normal. In this particular case, the PDF approaches Bernoulli, Y_n tends to approach Binomial, Binomial tends to Normal.

8.9
- Show that $E(|X_i - \mu_i|^3) = \left(1 - \left(\frac{1}{2}\right)^i\right)^3$

From our previous work, we showed that the means were zero, so this simplifies to the moment about the origin:

$$c_i = E(|X_i - \mu_i|^3) = \left|1 - \left(\frac{1}{2}\right)^i\right|^3 = \left(1 - \left(\frac{1}{2}\right)^i\right)^3$$

- Find A so that:

$$[\mathrm{var}(Y_n)]^{-3/2}\sum_{i=1}^{n} c_i = \frac{\sum_{i=1}^{n} A^3}{\left(\sum_{i=1}^{n} A^2\right)^{3/2}}$$

  You may assume that this fraction has the form:

$$\frac{n + \text{Terms go to zero as } n \to \infty}{(n + \text{Terms go to zero as } n \to \infty)^{3/2}}$$

  (And therefore, the CLT holds).
  Much of this is straight substitution with

$$A = 1 - \left(\frac{1}{2}\right)^i$$

8.16-17

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i^2 - 2X_i\bar{X} + \bar{X}^2\right) =$$

$$\frac{1}{n-1}\left(\sum_{i=1}^{n} X_i^2 - 2\bar{X}\sum_{i=1}^{n} X_i + \bar{X}^2\sum_{i=1}^{n} 1\right) =$$

$$\frac{1}{n-1}\sum_{i=1}^{n} X_i^2 - \frac{2}{n-1}\bar{X}n\bar{X} + \frac{n}{n-1}\bar{X} =$$

$$\frac{1}{n-1}\sum_{i=1}^{n} X_i^2 - \frac{n}{n-1}\bar{X}^2 =$$

$$\frac{n\sum X_i^2 - (n\bar{X})^2}{n(n-1)}$$

Calculate the sample variance of:

$$\{13, 14, 13, 11, 15, 14, 17, 11\}$$

$$\bar{X} = \frac{108}{8} = 13.5 \qquad \sum X_i^2 = 1486$$

Therefore,

$$S^2 = \frac{8 \cdot 1486 - (8 \cdot 13.5)^2}{8 \cdot 7} = \frac{11888 - 11664}{8 \cdot 7} = 4$$

8.63 Plug-n-Chug with parameters:

$$n = 100 \quad \mu = 75 \quad \sigma = 16 \qquad \sigma^2 = 256 = 16^2$$

- With Chebyshev first, noting that:

$$67 < \bar{X} < 83 \quad \Rightarrow \quad -8 < \bar{X} - \mu < 8 \quad |\bar{X} - \mu| < 5 \cdot 1.6$$

$$P(|\bar{X} - \mu| < 5 \cdot 1.6) \geq 1 - \frac{1}{5^2} = 0.96$$

- With Normal:

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{67 - 75}{1.6} = -5 \qquad \frac{83 - 75}{1.6} = 5$$

so that

$$P(67 < \bar{X} < 83) = 2P(0 < Z < 2) \approx 1.00$$

8.67 Uniform, $n = 200$, so $\mu = 36$ and $\sigma^2 = 48$.

$$\bar{X} < 35 \quad \Rightarrow \quad \frac{\bar{X} - \mu}{\sqrt{48}/\sqrt{200}} < -\frac{\sqrt{200}}{\sqrt{48}} \approx -2.04$$

This is $\frac{1}{2} - P(0 < Z < 2.04) = 0.5 - 0.4793 = 0.0207$

8.71 Assume that we have two normal populations, $n = 400$ from each, and

$$\mu_1 = \mu_2 = \mu \qquad \sigma_1 = 20 \qquad \sigma_2 = 30$$

From Exercise 8.3, the random variable $Y = \bar{X}_1 - \bar{X}_2$ is normal with mean $\mu_1 - \mu_2$ (zero in this exercise), and the variance is

$$\sigma_Y^2 = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} = \frac{20^2 + 30^2}{400} = \frac{13}{4}$$

Now,

$$-k < Y < k \quad \Rightarrow \quad -k < Y - \mu < k \quad \Rightarrow \quad -\frac{2}{\sqrt{13}}\,k < \frac{Y - \mu_y}{\sigma_y} < \frac{2}{\sqrt{13}}\,k$$

Find k so that the probability of being in the above interval is 0.99:

$$P\left(|Z| < \frac{2}{\sqrt{13}}\,k\right) = 0.99 \quad \Rightarrow \quad P\left(0 < Z < \frac{2}{\sqrt{13}}\,k\right) = 0.495$$

From the table, we see that:

$$\frac{2}{\sqrt{13}}\,k = 2.57 \quad \Rightarrow \quad k = 4.63$$