8.31 Show that the variance of the $t-$distribution is $\frac{\nu}{\nu-2}$ for $\nu > 2$. Hint: Use the substitution (with some different forms for later use):

$$\frac{1}{u} = 1 + \frac{t^2}{\nu} \qquad u = \frac{1}{1+\frac{t^2}{\nu}} \qquad \frac{1}{u} - 1 = \frac{t^2}{\nu} \qquad \frac{\nu(1-u)}{u} = t^2$$

The $t-$distribution is:

$$f(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\,\nu}\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}} = c\left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

We will compute $E(T^2)$,

$$E(T^2) = c\int_{-\infty}^{\infty} t^2\left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}\,dt$$

Notice that this is symmetric (even), so we can double the half integral,

$$E(T^2) = 2c\int_{0}^{\infty} t^2\left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}\,dt$$

We're going to make the suggested substitution. Notice that, as $t \to \infty$, $u \to 0$, and similarly, as $t \to 0$, $u \to 1$.

$$2c\int_{u=1}^{u=0} \frac{\nu(1-u)}{u} u^{\frac{\nu+1}{2}}\,dt$$

For dt, notice that:

$$-\frac{1}{u^2}\,du = \frac{2t}{\nu}\,dt \quad \Rightarrow \quad -\frac{\nu}{2u^2 t}\,du = dt$$

where

$$t = \frac{\nu^{1/2}(1-u)^{1/2}}{u^{1/2}}$$

Putting these two equations together, we get the following for dt:

$$dt = \frac{\nu}{2u^2}\cdot\frac{u^{1/2}}{\nu^{1/2}(1-u)^{1/2}}\,du = \frac{-\nu^{1/2}}{2u^{3/2}(1-u)^{1/2}}du$$

Put it all together, and incorporate the c:

$$\frac{2\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\,\nu}\Gamma\left(\frac{\nu}{2}\right)}\int_{u=1}^{u=0}\frac{\nu(1-u)}{u}u^{\frac{\nu+1}{2}}\cdot\frac{-\nu^{1/2}}{2u^{3/2}(1-u)^{1/2}}du$$

Simplify to get:

$$\nu\,\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\nu}{2}\right)}\int_0^1 u^{\frac{\nu-2}{2}-1}(1-u)^{\frac{3}{2}-1}\,du$$

Before continuing, **the Beta distribution is the missing piece** (p. 558)

$$f(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \qquad 0 < x < 1$$

Now, let's make our integral into a Beta: Notice that

$$\alpha = \frac{\nu-2}{2} \qquad \beta = \frac{3}{2} \qquad \alpha+\beta = \frac{\nu+1}{2}$$

and recall that (p. 202):

$$\Gamma(k) = (k-1)\Gamma(k-1)$$

Now, so far we have:

$$\nu \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\nu}{2}\right)} \int_0^1 u^{\frac{v-2}{2}-1}(1-u)^{\frac{3}{2}-1}\,du = \nu \frac{\Gamma(\alpha+\beta)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\nu}{2}\right)} \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,du$$

Hang on- We're just about done!

$$\Gamma\left(\frac{\nu}{2}\right) = \left(\frac{\nu}{2}-1\right)\Gamma\left(\frac{\nu}{2}-1\right) = \left(\frac{\nu}{2}-1\right)\Gamma(\alpha) = \frac{\nu-2}{2}\Gamma(\alpha)$$

And,

$$\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \Gamma\left(\frac{3}{2}\right) \quad \Rightarrow \quad \Gamma\left(\frac{1}{2}\right) = 2\Gamma\left(\frac{3}{2}\right) = 2\Gamma(\beta)$$

Therefore, the constant in front of the integral is:

$$\nu \frac{\Gamma(\alpha+\beta)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\nu}{2}\right)} = \nu \frac{\Gamma(\alpha+\beta)}{2\Gamma(\beta)\frac{\nu-2}{2}\cdot\Gamma(\alpha)} = \frac{\nu}{\nu-2}\cdot\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

Now the full integral for the $E(T^2)$ becomes:

$$\frac{\nu}{\nu-2}\int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1}(1-u)^{\beta-1}\,du = \frac{\nu}{\nu-2}$$

This is valid as long as $\alpha > 0$, or:

$$\frac{\nu-2}{2} > 0 \quad \Rightarrow \quad \nu > 2$$