Random Sampling of Questions

1. The probability that rain is followed by rain is 0.8, a sunny day is followed by rain is 0.6. Find the probability that one has two rainy days then two sunny days.

   To get this, we must have had rain after rain, $P(R|R)$, then sun after rain, $P(S|R)$, then rain after sun, $P(R|S)$. In the text, $P(R|R) = 0.8$, $P(R|S) = 0.6$. This gives $P(S|R) = 0.2$ (and $P(S|S) = 0.4$

   Therefore, $0.8 \cdot 0.2 \cdot 0.6 = 0.096$ or about 9.6%.

2. Let:
$$f(x) = k|x-2|, \quad \text{for} \quad x = -1, 0, 1, 3$$

   - Find k so that f is a PDF.
     We can build a table:

x	-1	0	1	3
$f(x)$	3	2	1	1
x^2	1	0	1	9

     Therefore, $k = 1/\sum f(x) = 1/7$

   - Find the expected value of X.
     Find $\sum x f(x) = \frac{-3+0+1+3}{7} = \frac{1}{7}$
   - Find the expected value of X^2
     Find $\sum x^2 f(x) = \frac{3+0+1+9}{7} = \frac{13}{7}$

3. A balanced die is tossed twice. Let A be the event that an even number comes up on the first toss. Let B be the event that an even number comes up on the second. Let C be the event that the first two tosses gave the same number. Are A, B, C

   - pairwise independent
   - independent?

   To verify our work, we could simply list the possibilities:

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

   From this, $P(A) = P(B) = \frac{1}{2}$ and $P(C) = \frac{1}{6}$.

   - Are A, B, C pairwise independent? To check, we will need $P(A \cap B) = 9/36 = 1/4$, $P(A \cap C) = 3/36 = 1/12$, $P(B \cap C) = 1/12$.
     Given these, it is easy to see that A, B, C are pairwise independent.
   - The three sets are not independent, however, since
$$P(A)P(B)P(C) = \frac{1}{24}$$
     But $A \cap B \cap C$ takes 3 out of 36, so the probability is 1/12.

4. State Bayes' Theorem.

   (Be able to draw the tree) Let $B_1, \ldots, B_k$ be a partition of the sample space. Then (I'm including intermediate steps that clarifies the final formula) for any event A (we assume $P(A) \neq 0$):

$$P(B_j|A) = \frac{P(A \cap B_j)}{P(A)} = \frac{P(A \cap B_j)}{\sum_{k=1}^{n} P(A \cap B_n)} = \frac{P(B_j)P(A|B_j)}{\sum_{n=1}^{k} P(B_n)P(A|B_n)}$$

5. If A, B are independent prove that A' and B' are independent.

   SOLUTION: (Exercise 2.22):

   In this case, show that

$$P(A' \cap B') = P(A')P(B')$$

   We might be able to do this straight off:

$$\begin{aligned}
P(A' \cap B') &= P((A \cup B)') && \text{M.E. sets} \\
&= 1 - P(A \cup B) \\
&= 1 - (P(A) + P(B) - P(A \cap B)) \\
&= 1 - P(A) - P(B) + P(A)P(B) && \text{Indep of } A, B \\
&= (1 - P(A)) - P(B)(1 - P(A)) \\
&= P(A')P(B')
\end{aligned}$$

6. What is the difference between saying that A, B, C are independent events, versus saying that X, Y, Z are independent random variables?

   To say that A, B, C are independent events, we must have that they are pairwise independent, $P(A \cap B) = P(A)P(B)$, for example, AND that $P(A \cap B \cap C) = P(A)P(B)P(C)$.

   To say that X, Y, Z are independent random variables, we must have that the joint PDF can be factored:

$$f(x, y, z) = f_1(x)f_2(y)f_3(z)$$

   where f_1, f_2, f_3 are the marginal distributions of X, Y, Z, respectively.

Extra Practice Problem: Let

$$f(x, y, z) = (x + y)\mathrm{e}^{-z}, \quad 0 < x < 1, 0 < y < 1, z > 0$$

and zero elsewhere. Below I've compute a bunch of marginal distributions:

- $g(x, z) = (x + \frac{1}{2})\mathrm{e}^{-z}$
- $g(x, y) = x + y$
- $g(y, z) = (y + \frac{1}{2})\mathrm{e}^{-z}$
- $g(x) = x + \frac{1}{2}$
- $g(y) = y + \frac{1}{2}$
- $g(z) = \mathrm{e}^{-z}$

Are X, Y, Z pairwise independent? If so, which? Are they independent (between all 3)?

SOLUTION: From the computations, we see that:

$$g(x, z) = g(x)g(z) \qquad g(y, z) = g(y)g(z) \qquad g(x, y) \neq g(x)g(y)$$

Therefore, X, Z are pairwise independent, Y, Z are pairwise independent, but X, Y are not. Furthermore, X, Y, Z are not independent (between all three), since

$$f(x,y,z) \neq g(x)g(y)g(z)$$

ADDITIONAL QUESTION: Is it possible to have three-way independence and NOT two-way independence? Consider that:

$$g(x,y) = \int_{-\infty}^{\infty} f(x,y,z)\,dz = g(x)g(y)\int_{-\infty}^{\infty} g(z)\,dz = g(x)g(y)$$

7. In a poker game, 5 cards are dealt from a deck of 52. What is the probability of getting a four-of-a-kind?

   Four of a kind means four 1's or four 2's, etc.

   Set this up as a sequence of operations:

   First, decide on what the four-of-a-kind will be: 13 choices.

   Next, how many ways can the four cards be chosen? 1 way

   Finally, how many ways can the last card be chosen? 48 ways.

   Thus, the probability is:

$$\frac{13 \cdot 48}{\binom{52}{5}} = \frac{1}{4165}$$

8. If the joint PDF is given by:

$$f(x,y,z) = \frac{xyz}{108} \text{ for } x = 1,2,3;\ y = 1,2,3;\ z = 1,2$$

   (a) Find the joint marginal distribution of X and Y.

$$g(x,y) = \frac{xy + 2xy}{108} = \frac{1}{36}xy$$

   (b) Find the conditional distribution of Z given $X = 1$ and $Y = 2$.
   First, the general distribution would be:

$$g(z|x,y) = \frac{g(x,y,z)}{g(x,y)}$$

   We found the denominator already, let's plug things in:

$$g(z|X=1,Y=2) = \frac{g(X=1,Y=2,Z)}{g(X=1,Y=2)} = \frac{\frac{2}{108}z}{\frac{6}{108}} = \frac{1}{3}z$$

   (c) Find the marginal distribution of X. $g(x) = \frac{x}{6}$. Notice that you can get this in a number of ways- For example, sum $g(x,y)$ over $y = 1,2,3$.

   (d) Are X, Y, Z independent random variables? Prove or Disprove.
   Yes they are. Compute the marginal distributions:

$$g(x) = \frac{x}{6} \qquad g(y) = \frac{y}{6} \qquad g(z) = \frac{z}{3}$$

   (Notice that $g(z) = g(z|x,y)$?) Then an easy computation shows that

$$f(x,y,z) = \frac{xyz}{108} = \frac{x}{6}\cdot\frac{y}{6}\cdot\frac{z}{3} = g(x)g(y)g(z)$$

9. How many ways can a college team playing 10 games end up with 5 wins, 4 losses and a tie?

   Think of a string of 10 slots that you can fill with 5 W's, 4 L's and 1 T:

$$\frac{10!}{5!4!1!} = 1260$$

10. A carton contains 15 light bulbs of which one is defective. In how many ways can you choose 3 of the bulbs and:

    (a) Get the one that is defective?

    To get the defective one, we had to choose 2 bulbs from the 14 good (then choose the defective): $\binom{14}{2}$

    (b) Not get the one that is defective?

    We would have to choose all three from the good bulbs, $\binom{14}{3}$

    (c) Find the probability of choosing the defective bulb?

    (Sorry this might not have been clear: We're still choosing three bulbs- otherwise, it would be 1/15!)

    The probability of choosing the defective bulb is the number of ways of selecting it, divided by the number of ways of selecting all the bulbs:

$$\frac{\binom{14}{2}}{\binom{15}{3}} = \frac{1}{5}$$

11. Given that 8% of the population has diabetes, a health department comes in and gives tests. It correctly diagnoses 95% of all persons with diabetes as having the disease, and incorrectly diagnoses 2% of all persons without diabetes as having the disease.

    (a) Find the probability that the health department will diagnose someone in the population as having the disease.

    (b) Find the probability that someone diagnosed by the health department as having the disease actually has it.

    SOLUTION: Some translations (Use these to label a tree diagram):

    - 8 percent of all adults over 50 have diabetes: $P(B) = 0.08$ (so $P(B') = 0.92$).
    - The health service correctly diagnoses 95 percent of all persons with the disease: $P(A|B) = 0.95$
    - It incorrectly diagnoses 2 percent of people without the disease as having it: $P(A|B') = 0.02$

    The first part of the question: What is the probability that the health service will diagnose blah blah blah: $P(A)$... Recall that in the tree diagram, sum down the right side:

$$P(A) = (0.08)(0.95) + (0.92)(0.02) = 0.0944$$

    In the second part, "A person diagnosed actually has the disease" is $P(B|A)$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)P(A|B)}{0.0944} = \frac{0.08 \cdot 0.95}{0.0944} = 0.805$$

12. Let $f(x) = \dfrac{1}{x\ln(3)}$ for $0 < x < 3$ (zero elsewhere). Find $E(X)$ and $E(X^2)$, then use the results to find $E(2X^2 - X + 1)$

TYPO: x cannot approach zero, otherwise the improper integral will not exist. Rather, we see the constant works if $1 < x < 3$:

$$\int_1^3 \frac{1}{x\ln(3)}\,dx = \frac{1}{\ln(3)}\left(\ln(x)\right|_1^3 = 1$$

Now,

$$E(X) = \int_1^3 \frac{x}{x\ln(3)}\,dx = \frac{2}{\ln(3)}$$

and

$$E(X^2) = \int_1^3 \frac{x^2}{x\ln(3)}\,dx = \frac{1}{\ln(3)}\int_1^3 x\,dx = \frac{4}{\ln(3)}$$

So that:

$$E(2X^2 - X + 1) = 2E(X^2) - E(X) + E(1) = 2\frac{4}{\ln(3)} - \frac{2}{\ln(3)} + 1$$

13. Experiment: Roll three standard dice. Observe the outcomes.

    (a) How many ways can the three dice all come up with the same number of points?
    (b) How many ways can two of the three come up with the same number, but the third die is different?
    (c) How many ways can all three dice come up with different numbers?

    SOLUTION: (Exercise 1.30):

    (a) One way for each triple. 6 ways.
    (b) If we fix the first, there are 5 ways for the other two dice to be the same (but different than the first). Thus, there are $6 \cdot 5 = 30$ ways.
    (c) To get all possible different valued triples, we could take: $6 \cdot 5 \cdot 4$. However, order does not matter, so divide by 3! to get 20.

14. Verify that $f(x) = \dfrac{2x}{k(k+1)}$ for $x = 1, 2, 3, \ldots k$ can serve as the probability distribution of a random variable with the given range.

    We need to recall that

$$\sum_{n=1^k} n = \frac{k(k+1)}{2}$$

15. State the Law of the Unconscious Statistician.

    If X is a random variable, with PDF f, then

$$E(g(X)) = \sum_x g(x)f(x) \qquad \text{or} \qquad E(g(X)) = \int_x g(x)f(x)\,dx$$

16. We have two men and four women making up a committee of three.

    - What is the probability of choosing no men? one man? two men?

$$\frac{\binom{2}{0}\binom{4}{3}}{\binom{6}{3}} = \frac{1}{5} \quad \frac{\binom{2}{1}\binom{4}{2}}{\binom{6}{3}} = \frac{3}{5} \quad \frac{\binom{2}{2}\binom{4}{1}}{\binom{6}{3}} = \frac{1}{5}$$

- Can you come up with a probability distribution function (PDF) for this using binomials? (see previous)

$$f(x) = \frac{\binom{2}{x}\binom{4}{3-x}}{\binom{6}{3}}, \quad x = 0, 1, 2$$

17. How many ways can a bakery distribute its 7 unsold apple pies to 4 food banks? How does this change if every food bank must get at least one?

    SOLUTION (1.55, 1.56)

    Think of the loaves of bread and bars problem from the homework. In this case, we have 7 A's (for apple pie) and 3 vertical bars. For example, $A|AA|AAA|A$ represents 1 pie to bank 1, 2 to bank 2, three to bank 3 and 1 to bank 4.

    This is just counting the ways of putting these symbols together:

    $$\binom{10}{7,3} = \frac{10!}{7!3!}$$

    If every food bank must get one pie, you can reserve 4 pies (one for each), and arrange the remaining three pies among the 4 food banks (again using three vertical bars):

    $$\binom{6}{3,3} = \frac{6!}{3!3!}$$

18. If I have four skirts, seven blouses and three sweaters, how many ways can I pick 2 skirts, three blouses and one sweater to take along on a trip?

    By the multiplication rule,

    $$\binom{4}{2} \cdot \binom{7}{3} \cdot \binom{3}{1} = 630$$

19. Two cards are randomly drawn from a deck of 52. Find the probability that both cards will be greater than 3 and less than 8.

    SOLUTION: Exercise 2.61

    First, how many cards represent a successful outcome? There are 16 cards (4-7 inclusive, each has 4 suits).

    Therefore, the probability is 16/52 for the first draw, and 15/51 for the second.

    Alternative: There are $\binom{16}{2}$ ways of success. Divide by $\binom{52}{2}$ total possible ways of choosing the two cards.

    (NOTE: Can you show that the two alternatives give the same result?)

20. If the joint density of X, Y is given by

    $$f(x,y) = \begin{cases} 4xy & \text{if } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

    (a) Compute $f(x|y)$ and $f(y|x)$.

    First, we see that:

    $$f(x|y) = \frac{f(x,y)}{g(y)} \qquad f(y|x) = \frac{f(x,y)}{g(x)}$$

so we need the marginal distributions of x and y:

$$g(x) = \int_0^1 f(x,y)\,dy = 4x\int_0^1 y\,dy = 2x \qquad g(y) = 2y$$

Therefore,

$$f(x|y) = \frac{4xy}{2y} = 2x \qquad f(y|x) = \frac{4xy}{2x} = 2y$$

Hmmm- We might have suspected that, since it looked like X and Y were going to be independent (see the last question).

(b) Set up an integral to determine $P(X+Y<1)$ In drawing this region, you should be looking at a square with a diagonal running from the upper left corner to the lower right corner- We're integrating over the lower half of the rectangle. One way to set it up:

$$\int_0^1 \int_{y=0}^{y=-x+1} 4xy\,dy\,dx = \int_0^1 2x(1-x)^2\,dx = \frac{1}{6}$$

(c) Find $F(x,y)$. Be careful of the regions (like we did in class).
If X, Y are inside the square, $0<x<1, 0<y<1$, then:

$$F(x,y) = \int_0^x \int_0^y 4st\,dt\,ds = x^2y^2$$

We can fill in the rest:

$$F(x,y) = \begin{cases} 0 & \text{if } x \le 0 \text{ or } y \le 0 \text{ (or both)} \\ x^2y^2 & \text{if } 0<x<1 \text{ and } 0<y<1 \\ y^2 & \text{if } x>1 \text{ and } 0<y<1 \\ x^2 & \text{if } y>1 \text{ and } 0<x<1 \\ 1 & \text{if } x \ge 1 \text{ and } y \ge 1 \end{cases}$$

(d) Are X, Y independent random variables?

(e) Set up an integral to determine $E(X)$ and $E(Y)$. Is the product the same as $E(XY)$?
Yes:

$$E(XY) = \int_0^1 \int_0^1 4x^2y^2\,dx\,dy = \int_0^1 2x^2\,dx \cdot \int_0^1 2y^2\,dy = E(X)E(Y)$$