

Extra Counting Practice

1. A fair coin is tossed 10 times. (i) Find the probability of getting exactly three heads. (ii) Find the probability of getting 3 or fewer heads.

SOLUTION: We set it up by counting how many things are in the sample space, then count how many ways of getting the three heads.

For each of the 10 coin tosses, we have either a head (H) or a tail (T). Therefore, there are 2^{10} strings in the sample space (and each is equally likely).

To count how many ways we get exactly three heads, we might think about the number of ways of selecting three out of 10 slots- Those we will take to be the H, the rest will be T. This is $\binom{10}{3}$. Overall,

$$\frac{\binom{10}{3}}{2^{10}} \approx 0.117$$

From this, the solution to part (ii) will be:

$$\frac{\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3}}{2^{10}} \approx 0.17$$

2. A class consists of 15 boys and 30 girls. Ten students are selected at random for a special committee. Find the probability that exactly three boys are selected.

SOLUTION: First, how many “objects” are in the sample space? We are choosing 10 students from the 45, so: $\binom{45}{10}$.

Now we need three boys, and the number of ways of selecting them will be $\binom{15}{3}$.

If we could not tell the girls apart, we’d be done- But of course we can choose them apart. Therefore, we also need to account for the number of ways we can put 7 girls on the committee, which is $\binom{30}{7}$.

Altogether:

$$\frac{\binom{15}{3} \binom{30}{7}}{\binom{45}{10}} \approx 0.29$$

3. We will deal all the cards of a standard deck (52 cards) to 4 players (so each get 13 cards). Find the probability that each player gets (exactly) one ace.

We might start by thinking about stringing the 52 cards all out in a line- The first player would get the first string of 13, the next player would get the second string of 13, and so on.

Since the question deals only with the number of aces in each hand and not the overall hands, we are only interested in the different positions that the aces might have in the string of 52 cards. Therefore, the sample space contains

$$\binom{52}{4}$$

ways of placing the 4 aces in the 52 card “slots”.

Now, if each player has exactly one ace, then each player had 13 ways of getting that ace. Therefore, there are 13^4 ways for all players to get one ace.

Putting it together, the probability that each player gets exactly one ace is:

$$\frac{13^4}{\binom{52}{4}} = 0.1055$$

ALTERNATIVE SOLUTION: As an extreme example of an alternative solution, let us differentiate between orderings of the 52 cards. In this case, we will also need to differentiate between orderings of the four aces- Let's see if we get the same answer.

Distinguishing the order of the cards, there are $52!$ ways to arrange the 52 cards (that is the number of things in the sample space).

As before, there are 13^4 ways to choose the four aces so that each player has one. There are $4!$ ways of ordering the four aces. That leaves $48!$ ways of ordering the other cards.

Altogether, that leaves us with the probability:

$$\frac{13^4 \times 4! \times 48!}{52!} = 0.1055$$

Which is easier? Probably the first way- But this shows us that there are multiple ways of solving the problem. Whichever way we choose, we need to stay consistent.

4. A box has 24 light bulbs, where exactly 2 are defective. If you select 10 bulbs (without replacement) what is the probability that both bad bulbs will be selected?

We are selecting 10 bulbs from 24, so there are $\binom{24}{10}$ things in the sample space.

We now need to select the 2 bad bulbs from 2 possible choice, $\binom{2}{2}$, and we need to select 8 good bulbs from the 22, $\binom{22}{8}$.

Put it together so the probability of selecting both bad bulbs is:

$$\frac{\binom{2}{2} \binom{22}{8}}{\binom{24}{10}} \approx 0.16$$

(Also see the defective TV's, Exercise 1.49, p. 21)

5. A boy has 5 blue and 4 white marbles in his left pocket, and 4 blue, 5 white marbles in his right pocket. If he transfers one marble from his left pocket to his right, what is the probability of his then drawing a blue marble from his right pocket?

This question originally had a typo that should be corrected now- The first transfer should be from the left to the right, not vice versa.

Before drawing the blue marble from his right pocket, he transferred either a blue marble or a white marble from his left pocket.

The probability of transferring a blue marble is $5/9$. He then has 5 blue and 5 white marbles in his right pocket, so the probability of drawing blue is $5/10$. All told, the probability of this event occurring is $\frac{5}{9} \cdot \frac{5}{10}$

The other case was transferring a white marble, the probability of which is $4/9$. The boy then has 4 blue and 6 white marbles in his right pocket, so the probability of drawing blue is $4/10$. The overall probability of this event is $\frac{4}{9} \cdot \frac{4}{10}$

The overall probability of drawing blue from his right pocket:

$$\frac{5}{9} \cdot \frac{5}{10} + \frac{4}{9} \cdot \frac{4}{10} = \frac{41}{90} \approx 0.46$$

6. In a small pond, there are 50 fish where 10 of them have been tagged. A fisherman selects seven fish at random and without replacement. Find the probability that 2 of the tagged fish have been caught.

This is similar to the defective TV's and the defective bulbs. In this case, we are choosing 2 tagged fish (out of 10), and 5 untagged fish (out of 40):

$$\frac{\binom{10}{2} \binom{40}{5}}{\binom{50}{7}} \approx 0.296$$

7. From a 1995 study of the wreck of the *Titanic*, it was found that, of the adults on board:

	Male	Female	Totals
Survived	338	316	654
Died	1329	109	1438
Totals	1667	425	2092

(NOTE: It is most helpful if you think about how to convert each question into a mathematical statement before computing the answers)

For extra practice, we translate these quantities using S, D, M, F as the events survived, died, male, female respectively. To facilitate the answers, we also provided totals to the table above.

- (a) What was the probability of dying, given that you were female (assume only adults)?

$$P(D|F) = \frac{P(D \cap F)}{P(F)} = \frac{109}{425} \approx 0.26$$

(NOTE: Formally, both the numerator and denominator would be divided by the overall total- Which you can do if you would like).

- (b) What was the probability of being an adult male on the *Titanic* on the day she went down?

$$P(M) = \frac{1667}{2092} \approx 0.77$$

- (c) What was the probability of dying on the *Titanic* (from the adult data)?

$$P(D) = \frac{1438}{2092} \approx 0.69$$

- (d) Would the number $\frac{1329}{1329 + 109}$ be meaningful?

Using the events we defined, we would interpret this number as:

$$\frac{P(M \cap D)}{P(D)} = P(M|D)$$

This is the probability of being male given the dead (about 92%).

Is this being male and dying? No If not, what is that probability? This is the intersection (the union would use “or” instead of “and”):

$$P(M \cap D) = \frac{1329}{2092} \approx 0.64$$

8. Does a monkey have a better chance of rearranging

$A C C L L U U S$ to spell $CALCULUS$ or $A A B E G L R$ to spell $ALGEBRA$

We know that the total number of ways of arranging the letters in $CALCULUS$ are:

$$\frac{8!}{1!2!2!2!1!} = 7! = 5,040$$

So there is one way out of 5,040, or a probability of about 0.01 %

And the ways of arranging the letters in $ALGEBRA$ are:

$$\frac{7!}{2!1!1!1!1!} = 2,520$$

So the probability of getting the word $ALGEBRA$ is $1/2520 \approx 0.03\%$

Therefore, it is more likely to rearrange the letters of $ALGEBRA$ to get that string than it is $CALCULUS$.

9. Nine cards are dealt from a standard 52-card deck. Write a formula for the probability that three of the five even numerical denominations (those are 2, 4, 6, 8, 10) are represented twice, one of the three face cards (those are Jack, Queen, King) appears twice, and a second face card appears once. HINT: It's important to piece this one out.

SOLUTION:

There are $\binom{52}{9}$ objects in the sample space.

To get three even numbers to appear twice, we'll need to choose which of them we'll use ($\binom{5}{3}$ ways to do that), then for each value (2, 4, 6, 8 or 10), there are $\binom{4}{2}$ ways of selecting them (for example, a 2 of hearts and a 2 of clubs). So far we have

$$\binom{5}{3} \binom{4}{2}^3$$

ways of getting the even cards. Next we want one face card to appear twice. There are $\binom{3}{1}$ ways of choosing the value of the face card, and $\binom{4}{2}$ ways of selecting the cards.

There are 2 face card values remaining, and we pick one value. There are $\binom{4}{1}$ ways of selecting the card with that face value.

Altogether, the probability of our event is:

$$\frac{\binom{5}{3} \binom{4}{2}^3 \binom{3}{1} \binom{4}{2} \binom{2}{1} \binom{4}{1}}{\binom{52}{9}}$$

10. Doug is not the world's greatest poker player. Dealt a two of diamonds, an 8 of diamonds, an ace of hearts, an ace of clubs and an ace of spades, he discards the three aces. What are the chances of drawing to a flush? You may assume that Doug is the only one getting cards from that deck. (A flush is where the cards are all of the same suit, but not necessarily in numerical order).

We'll be choosing three cards from the 47 cards remaining, there are $\binom{47}{3}$ ways of doing that. We have two cards that are diamonds already, so we need to get three more from the 11 diamond cards remaining (there are $\binom{11}{3}$ ways to get those):

$$\frac{\binom{11}{3}}{\binom{47}{3}}$$