Expectation and Independence

Let $f(x,y) = \frac{xy}{9}$ for $x = 1,2; y = 1,2$.

- Are X, Y independent random variables?

  By taking $f_1(x) = \frac{x}{3}$ and $f_2(y) = \frac{y}{3}$, we see that, each of these individually are pdfs, and the joint pdf is:

  $$f(x,y) = f_1(x)f_2(y)$$

- What is $E(X)$, $E(Y)$ and $E(XY)$?

  $$E(X) = \sum_{x=1}^{2}\sum_{y=1}^{2}\frac{x^2}{3}\cdot\frac{y}{3} = \sum_{x=1}^{2}\frac{x^2}{3}\left(\sum_{y=1}^{2}\frac{y}{3}\right) = \frac{5}{3}\cdot 1 = \frac{5}{3}$$

  Notice that this is the same $E(X)$ as if we had only looked at $f_1(x) = x/3$. That is, the expected value of X in the joint distribution is the same as that in the individual distribution (of course, this is ONLY true for independent variables).

  Similarly, $E(Y) = 5/3$ To compute $E(XY)$, we have:

  $$\sum_{x=1}^{2}\sum_{y=1}^{2} xyf(x,y) = \sum_{x=1}^{2}\sum_{y=1}^{2}\frac{1}{9}x^2y^2 = \frac{1}{9}(1+4+4+16) = \frac{25}{9}$$

- In the discrete case,

  $$E(XY) = \sum_x\sum_y xyf(x,y) = \sum_x\sum_y xf_1(x)yf_2(y) = \sum_x xf_1(x)\left(\sum_y yf_2(y)\right)$$

  This last term is independent of x, and can be factored out:

  $$E(XY) = \left(\sum_x xf_1(x)\right)\left(\sum_y yf_2(y)\right) = E(X)E(Y)$$

- If X and Y are independent random variables, then $E(XY) = E(X)E(Y)$.

  Start with the definition, then use independence:

  $$E(XY) = \iint xyf(x,y)\,dx\,dy = \iint xyf_1(x)f_2(y)\,dx\,dy =$$

  $$\iint xyf_1(x)f_2(y)\,dx\,dy = \int xf_1(x)\,dx\int yf_2(y)\,dy = E(X)E(Y)$$