Quiz 3 SOLUTIONS

1. Let $f(x, y, z) = kxyz$ with $x \in \{1, 2\}$, $y \in \{1, 2, 3\}$, and $z \in \{1, 2\}$ (and f is zero elsewhere).

   (a) Find k so that f is a probability distribution function.
   SOLUTION: The sum over all input should be 1. Factoring k out,

   $z = 1 \Rightarrow$

	$x = 1$	$x = 2$
$y = 1$	1	2
$y = 2$	2	4
$y = 3$	3	6

   $z = 2 \Rightarrow$

	$x = 1$	$x = 2$
$y = 1$	2	4
$y = 2$	4	8
$y = 3$	6	12

   The full sum is 54, so $k = 1/54$.

   (b) Find $P(X = 1, Y + Z = 3)$:

   $$= P(X = 1, Y = 1, Z = 2) + P(X = 1, Y = 2, Z = 1) = \frac{1 \cdot 1 \cdot 2 + 1 \cdot 2 \cdot 1}{54} = \frac{2}{27}$$

   (c) If F is the cumulative distribution, what is $F(2, 1, 2)$? There are two lines in the chart above corresponding to $Y = 1$, and X, Z can be in their full range:

   $$\frac{1 + 2 + 2 + 4}{54} = \frac{9}{54} = \frac{1}{6}$$

   (d) Find the *marginal distribution*, $h(y)$. Summing across the tables,

   $$h(1) = \frac{1 + 2 + 2 + 4}{54} = \frac{1}{6} \qquad h(2) = \frac{2 + 4 + 4 + 8}{54} = \frac{1}{3} \qquad h(3) = \frac{3 + 6 + 6 + 12}{54} = \frac{27}{54} = \frac{1}{2}$$

2. Let the joint distribution function be given by:

   $$F(x, y) = 1 - e^{-x} - e^{-y} + e^{-x-y}$$

   if $x > 0, y > 0$ (zero elsewhere).

   (a) Compute $P(1 < X < 2, 1 < Y < 2)$ This is $F(2, 2) = F(1, 2) - F(2, 1) + F(1, 1)$

   $$e^{-2} + e^{-4} - 2e^{-3} \approx 0.054$$

   (b) Find the joint probability density, f.

   $$f = F_{xy} \qquad F_x = e^{-x} - e^{-x}e^{-y} \qquad F_{xy} = e^{-(x+y)}$$

   And this is valid in Quadrant I (zero elsewhere).

   (c) Find the marginal cumulative distribution,

   $$H(y) = P(Y \le y) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} f(x, t)\, dx\, dt = F(\infty, y) = 1 - e^{-y}$$

3. A joint PDF is given below.

   $$f(x, y) = xye^{-(x+y)}, \quad x > 0, \quad y > 0$$

   Find $F(s, t)$, the cumulative distribution function.

   Integration by parts note: We can build a table to compute $\int xe^{-x}\, dx$

sign	u	dv
$+$	x	e^{-x}
$-$	1	$-e^{-x}$
$+$	0	e^{-x}

   $$\Rightarrow \quad \int xe^{-x}\, dx = -xe^{-x} - e^{-x} = -e^{-x}(x + 1)$$

   Now,

   $$F(s, t) = \int_{-\infty}^{s} \int_{-\infty}^{t} xye^{-(x+y)}\, dy\, dx = \int_{-\infty}^{s} \int_{-\infty}^{t} xe^{-x}ye^{-y}\, dy\, dx = \left(\int_{-\infty}^{s} xe^{-x}\, dx\right) \left(\int_{-\infty}^{t} ye^{-y}\, dy\right)$$

   $$\left(e^{-s}(s + 1) - 1\right) \left(e^{-t}(t + 1) - 1\right)$$