Review Questions, Final Exam

1. General questions:

   (a) What is the Fundamental Theorem of Linear Programming?
   SOLUTION: We can paraphrase this since it is not given in the text this way, but the main point is that when an LP has an optimal solution, it has an optimal BFS (which means we only need to search through the BFS).

   (b) What is the main idea behind the Simplex Method? (Think about what it is doing graphically- How does the algorithm start, how does it proceed?)
   SOLUTION: Actually, I included the main idea in the first answer- We begin with a BFS, and determine if that BFS is optimal. If it is not, then we determine which adjacent BFS should come into the algorithm next. If we cannot bring any BFS in, then the algorithm terminates.

2. Consider the following LP:

$$\begin{aligned} \min z = \quad & 3x - 4y + 2z \\ \text{st} \quad & 2x - 4y \geq 4 \\ & x + z \geq -5 \\ & y + z \leq 1 \\ & x + y + z = 3 \end{aligned}$$

with $x \geq 0, y$ is URS, $z \geq 0$.

   (a) Write the dual.
   SOLUTION: Sorry for using x, y, z for the primal- Maybe use u_i for the dual- In that case,

$$\begin{aligned} \max w = \quad & 4u_1 - 5u_2 + u_3 + 3u_4 \\ \text{st} \quad & 2u_1 + u_2 + u_3 + u_4 \leq 3 \\ & -4u_1 + u_3 + u_4 = -4 \\ & u_2 + u_3 + u_4 \leq 2 \end{aligned}$$

   with $u_1 \geq 0$, $u_2 \geq 0$, $u_3 \leq 0$, and u_4 URS.

   (b) Going back to the original LP, write it in standard form as a max problem with equality constraints, and then write the initial tableau (before big-M or other methods).
   SOLUTION: To prepare for the simplex method, we'll make this a max problem and get rid of the -5 on the right hand side of a constraint. Further, since y is URS, replace $y = y_1 - y_2$, where $y_{1,2} \geq 0$. Finally, add excess or slack variables to turn the inequalities into equality constraints.

$$\begin{aligned} \min z = \quad & 3x - 4y + 2z \\ \text{st} \quad & 2x - 4y \geq 4 \\ & x + z \geq -5 \\ & y + z \leq 1 \\ & x + y + z = 3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max z = \quad & -3x + 4(y_1 - y_2) - 2z \\ \text{st} \quad & 2x - 4(y_1 - y_2) - e_1 = 4 \\ & -x - z + s_1 = 5 \\ & (y_1 - y_2) + z + s_2 = 1 \\ & x + (y_1 - y_2) + z = 3 \end{aligned}$$

   Therefore, the tableau is:

x	y_1	y_2	z	e_1	s_1	s_2	rhs
3	−4	4	2	0	0	0	0
2	−4	4	0	−1	0	0	4
−1	0	0	−1	0	1	0	5
0	1	−1	1	0	0	1	1
1	1	−1	1	0	0	0	3

3. Consider again the "Wyndoor" company example we looked at in class:

$$\begin{aligned} \min z = \ & 3x_1 + 5x_2 \\ \text{st} \ & x_1 \le 4 \\ & 2x_2 \le 12 \\ & 3x_1 + 2x_2 \le 18 \end{aligned}$$

with x_1, x_2 both non-negative.

(a) SOLUTION:
Define the extra variables s_1, s_2, s_3.
Using the extra variables in order, the constraints become:

$$\begin{array}{cccccc} x_1+ & & s_1 & & & = 4 \\ & 2x_2+ & & s_2 & & = 12 \\ 3x_1+ & 2x_2+ & & & s_3 & = 18 \end{array}$$

And from this, it is easy to read off the coefficient matrix A.

(b) Is the following a basic solution? Is it a basic feasible solution?

$$x_1 = 0, x_2 = 6, s_1 = 4, s_2 = 0, s_3 = 6$$

Which variables are BV, and which are NBV?
SOLUTION: The matrix A has rank 3. If the solution has $n - m = 5 - 3 = 2$ zeros (and it is a solution), then it is a basic solution: Yes, this is a basic solution. It is also a basic *feasible* solution since every entry of the basic solution is non-negative. The variables x_2, s_1 and s_3 are the basic variables (BV) and the variables x_1 and s_2 are NBV.

(c) Find the basic feasible solution obtained by taking s_1, s_3 as the non-basic variables.
In this case, we can row reduce the augmented matrix (remove columns 3 and 5 from the original):

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 2 & 1 & 12 \\ 3 & 2 & 0 & 18 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

In this case, we have the (full) solution:

$$x_1 = 4, \quad x_2 = 3, \quad x_3 = 0, \quad x_4 = 6, \quad x_5 = 0$$

4. Given the current tableau (with variables labeled above the respective columns), answer the questions below.

$$\begin{array}{rrrr|r} x_1 & x_2 & s_1 & s_2 & \text{rhs} \\ 0 & -1 & 0 & 2 & 24 \\ \hline 0 & 1/3 & 1 & -1/3 & 1 \\ 1 & 2/3 & 0 & 1/3 & 4 \end{array}$$

(a) Is the tableau optimal (and did your answer depend on whether we are maximizing or minimizing)? For the remaining questions, you may assume we are maximizing.
ANSWER: This tableau is not optimal for either. If we were minimizing, we could still pivot using s_2. If we were maximizing, we could still pivot in x_2.

(b) Give the current BFS.
ANSWER: The current BFS is $x_1 = 4, x_2 = 0, s_1 = 1$ and $s_2 = 0$.

(c) Directly from the tableau, can I increase x_2 from 0 to 1 and remain feasible? Can I increase it to 4?

ANSWER: From the ratio test, x_2 can be increased to 3 in the first, and 6 in the second. However, increasing it to 4 would violate the first constraint. Summary: I can increase x_2 from 0 to 1, but not to 4.

(d) If x_2 is increased from 0 to 1, compute the new value of z, x_1, s_1 (assuming s_2 stays zero).

SOLUTION:

$$z = 25 \qquad x_1 = \frac{10}{3} \qquad s_1 = \frac{2}{3}$$

(e) Write the objective function and all variables in terms of the non-basic (or free) variables, and then put them in vector form.

SOLUTION: For the current tableau, $z = 24 + x_2 - 2s_2$, with

$$\begin{aligned} x_1 &= 4 - 2/3x_2 - 1/3s_2 \\ x_2 &= \qquad\quad x_2 \\ s_1 &= 1 - 1/3x_2 + 1/3s_2 \\ s_2 &= \qquad\qquad\quad s_2 \end{aligned} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{x_2}{3}\begin{bmatrix} -2 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \frac{s_2}{3}\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

5. Given the following final tableau, find two solutions to the original problem.

x_1	x_2	x_3	s_1	s_2	rhs
0	0	5	0	1	15
0	2/5	9/5	1	−1/5	3
1	3/5	6/5	0	1/5	3

SOLUTION: We want to interpret this final tableau. Notice that:

$$z = 15 - x_3 - s_2$$

which does not depend on x_2 (and note that the Row 0 coefficient of $x_2 = 0$). Therefore, x_2 could be pivoted in for another BFS. However, we also know that the current system of equations has the solution:

$$\begin{aligned} x_1 &= 3 - 3/5x_2 \\ x_2 &= x_2 \\ x_3 &= 0 \\ s_1 &= 3 \\ s_2 &= 0 \end{aligned}$$

Anything along this line is also a solution (as long as $x_1, x_2 \geq 0$).

3. Set up the initial tableau for the big-M method, and state what your first step would be.

$$\begin{aligned} \max z = \quad & 5x_1 - x_2 \\ \text{st} \quad & 2x_1 + x_2 = 6 \\ & x_1 + x_2 \leq 4 \\ & x_1 + 2x_2 \leq 5 \end{aligned}$$

with $x_1, x_2 \geq 0$.

SOLUTION: This LP becomes:

$$\begin{array}{rl} \max z = & 5x_1 - x_2 - Ma_1 \\ \text{st} & 2x_1 + x_2 + a_1 = 6 \\ & x_1 + x_2 + s_1 = 4 \\ & x_1 + 2x_2 + s_2 = 5 \end{array} \quad \Rightarrow \quad \begin{array}{ccccc|c} x_1 & x_2 & a_1 & s_1 & s_2 & \text{rhs} \\ -5 & 1 & M & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 & 6 \\ 1 & 2 & 0 & 1 & 0 & 4 \\ 1 & 2 & 0 & 0 & 1 & 5 \end{array}$$

Our first step is to use row reduction to put a zero under the a_1 variable (so that the column for a_1 is the first column of the identity matrix).

6. Suppose we have obtained the tableau below for a maximization problem. State conditions on $a_1, a_2, a_3, b, c_1, c_2$ that are required to make the following statements true:

   (a) The current solution is optimal, and there are alternative optimal solutions.
   SOLUTION: $b \geq 0$ is necessary. If $c_1 = 0$ and $c_2 \geq 0$, we can pivot in x_1 for an alternate solution. If $c_1 \geq 0$, $c_2 \geq 0$ and $a_2 > 0$, we can pivot in x_5 and obtain an alternate solution. If $c_2 = 0$, $a_1 > 0$ and $c_1 \geq 0$, we can pivot in x_2 and get an alternate solution.

   (b) The current basic solution is not a BFS. SOLUTION: $b < 0$.

   (c) The current basic solution is a degenerate BFS. SOLUTION: $b = 0$

   (d) The current basic solution is feasible, but the LP is unbounded.
   SOLUTION: $b \geq 0$ makes the current solution feasible. If $c_2 < 0$ and $a_1 \leq 0$, we can make x_2 as large as desired (unbounded).

   (e) The current basic solution is feasible, but the objective function can be improved by replacing x_6 with x_1 as a basic variable.
   SOLUTION: $b \geq 0$ makes the current solution feasible. For x_6 to replace x_1, we need $c_1 < 0$, and we need Row 3 to win the ratio test for x_1. This means that $3/a_3 \leq b/4$.

x_1	x_2	x_3	x_4	x_5	x_6	rhs
c_1	c_2	0	0	0	0	10
4	a_1	1	0	a_2	0	b
-1	-5	0	1	-1	0	2
a_3	-3	0	0	-4	1	3

7. (This is from our Group Work Handout for Chapter 6)

   In the textbook's "Dakota Problem", we are making desks, tables and chairs, and we want to maximize profit given constraints on lumber, finishing and carpentry (resp).

   For the primal, let x_1, x_2, x_3 be the number of desks, tables and chairs we make (resp), where the original (max) tableau is as given below:

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & rhs \\ -60 & -30 & -20 & 0 & 0 & 0 & 0 \\ \hline 8 & 6 & 1 & 1 & 0 & 0 & 48 \\ 4 & 2 & \frac{3}{2} & 0 & 1 & 0 & 20 \\ 2 & \frac{3}{2} & \frac{1}{2} & 0 & 0 & 1 & 8 \end{array} \quad \rightarrow \quad \begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & rhs \\ 0 & 5 & 0 & 0 & 10 & 10 & 280 \\ \hline 0 & -2 & 0 & 1 & 2 & -8 & 24 \\ 0 & -2 & 1 & 0 & 2 & -4 & 8 \\ 1 & \frac{5}{4} & 0 & 0 & -\frac{1}{2} & \frac{3}{2} & 2 \end{array}$$

(a) Write the down vectors/matrices that we typically use in our computations. Namely, $\mathbf{c}, \mathbf{c}_B, B$, and B^{-1}.

$$\mathbf{c} = \begin{bmatrix} 60 \\ 30 \\ 20 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{c}_B = \begin{bmatrix} 0 \\ 20 \\ 60 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 8 \\ 0 & \frac{3}{2} & 4 \\ 0 & \frac{1}{2} & 2 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -1/2 & 3/2 \end{bmatrix}$$

(b) Using our vector notation, if $\mathcal{B}$ gives the optimal basis, how do we compute the dual, $\mathbf{y} = (\mathbf{c}_B^T B^{-1})^T$ (the transpose makes it a column).

(c) Write down the dual (either as an initial tableau or in "normal form").

In "normal form",

$$\begin{aligned} \min\ w \quad &= 48y_1 + 20y_2 + 8y_3 \\ \text{st} \quad & 8y_1 + 4y_2 + 2y_3 \geq 60 \\ & 6y_1 + 2y_2 + \tfrac{3}{2}y_3 \geq 30 \\ & y_1 + \tfrac{3}{2}y_2 + \tfrac{1}{2}y_3 \geq 20 \\ & \mathbf{y} \geq 0 \end{aligned}$$

(d) Using the optimal Row 0 from the primal, write down the solution to the dual:

$$\mathbf{y} = [0, 10, 10]^T$$

(e) In our "normal form", we have $A\mathbf{x} \leq \mathbf{b}$ for the primal and $A^T\mathbf{y} \geq \mathbf{c}$ for the dual.

- The "slack" for the primal, given $\mathbf{x}$: $\mathbf{b} - A\mathbf{x}$:
SOLUTION: These are s_1, s_2, s_3, which are already solved for:

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 0 \\ 0 \end{bmatrix}$$

- The "slack" for the dual, given $\mathbf{y}$: $A^T\mathbf{y} - \mathbf{c}$:
SOLUTION: These are e_1, e_2, e_3 for the dual, which we can get from (3):

$$\begin{aligned} 8(0) + 4(10) + 2(10) - e_1 &= 60 \\ 6(0) + 2(10) + 1.5(10) - e_2 &= 30 \\ (0) + (1.5)(10) + 0.5(10) - e_3 &= 20 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

You might notice there's a vector just like this in the final Row 0 of the primal!

(f) What is the shadow price for each constraint?

SOLUTION: The shadow prices are the solutions to the dual, $0, 10, 10$ for constraints 1, 2, and 3, respectively.

(g) Write down the inequalit(ies) we need for Δ if we change the coefficient of x_2 from 30 to $30 + \Delta$, and we want the current basis to remain optimal.

SOLUTION: Since x_2 is a NBV, we can compute this by either: $5 - \Delta > 0$ so $\Delta < 5$, or by using the second constraint of the dual:

$$6(0) + 2(10) + 1.5(10) \geq 30 + \Delta \quad \Rightarrow \quad 5 \geq \Delta$$

(h) Write down the inequalit(ies) we need for Δ if we change the coefficient of x_3 from 20 to $20 + \Delta$, and we want the current basis to remain optimal.

SOLUTION: This is a change to a NBV, so we take:

$$\begin{array}{rcccccc} \text{Old Row 0}: & X & 5 & X & X & 10 & 10 \\ +(\Delta)(\text{Row 2}): & X & -2\Delta & X & X & 2\Delta & -4\Delta \\ \hline & 0 & 5-2\Delta & 0 & 0 & 10+2\Delta & 10-4\Delta \end{array} \qquad \Rightarrow \qquad -5 < \Delta < \frac{5}{2}$$

(i) How does changing a *column* of A effect the dual? Use this to see what would happen if we change the column for x_2 (tables) to be $[5, 2, 2]^T$- Is it now worth it to make tables?

SOLUTION: Changing the 2d column of A means changing the 2d constraint for the dual, so we'll go ahead and check that, using our current solution to the dual:

$$5(0) + 2(10) + 2(10) \geq 30? \quad \Rightarrow \quad \text{Yes.}$$

Interpretation: The dual is still feasible, so therefore, the current basis for the primal is still optimal. That means we should NOT bring in x_2 (keep x_2 at 0).

(j) How does creating a new column of A effect the dual? Use this to see if it makes sense to manufacture footstools, where we sell them for \$15 each, and the resources are $[1, 1, 1]^T$.

SOLUTION: Adding a new column (or activity) in the primal corresponds to adding a new constraint to the dual:

$$1(0) + 1(10) + 1(10) \geq 15? \quad \Rightarrow \quad \text{Yes.}$$

Therefore, the dual is still feasible, and the current basis for the primal is still optimal. We should not make any footstools at this price (we see that we would need to price them at at least \$20 each).

8. Consider the LP and the optimal tableau with missing Row 0 and missing optimal RHS (assume big-M).

$$\begin{array}{rl} \max z = & 3x_1 + x_2 \\ \text{s.t.} & 2x_1 + x_2 \leq 4 \\ & 3x_1 + 2x_2 \geq 6 \\ & 4x_1 + 2x_2 = 7 \\ & x_1, x_2 \geq 0 \end{array}$$

x_1	x_2	s_1	e_2	a_2	a_3	rhs
0	0	1	0	0	$-1/2$	
0	1	0	-2	2	$-3/2$	
1	0	0	1	-1	1	

Find Row 0 and the RHS for the optimal tableau (without performing row reductions!)

SOLUTION: Row 0 coefficients are given by

$$-\mathbf{c}^T + \mathbf{c}_B^T B^{-1} A \qquad \text{where} \qquad \mathbf{c}^T = [3, 1, 0, 0, -M, -M] \quad \text{and} \quad \mathbf{c}_B^T = [0, 1, 3]$$

The columns in the tableau are already the columns for $B^{-1}A$, so we can use them. Here are the Row zero coefficients:

- For e_2: $[0, 1, 3][0, -2, 1]^T = 1$
- For a_2: $M + [0, 1, 3][0, 2, -1] = -1 + M$
- For a_3: $M + [0, 1, 3][-1/2, -3/2, 1]^T = 3/2 + M$
- Optimal z is $[0, 1, 3][1/2, 3/2, 1]^T = 9/2$
- We have zeros for x_1, x_2, s_1.

Therefore, the optimal tableau is:

x_1	x_2	s_1	e_2	a_2	a_3	rhs
0	0	0	1	$-1+M$	$3/2+M$	9/2
0	0	1	0	0	$-\frac{1}{2}$	$\frac{1}{2}$
0	1	0	-2	2	$-\frac{3}{2}$	$\frac{3}{2}$
1	0	0	1	-1	1	1

9. Give an argument why, if the primal is unbounded, then the dual must be infeasible.

   SOLUTION:

   Suppose the dual was feasible. Then there exists a $\mathbf{y}$ that satisfies the constraints for the dual. But in that case, for every $\mathbf{x}$ in the primal,

   $$\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$$

   But we said that the primal was unbounded, so that $\mathbf{c}^T\mathbf{x} \to \infty$. This is a contradiction- Thus, the dual must be infeasible.

10. Consider the following LP and its optimal tableau, shown.

$$\begin{aligned} \max z = \quad & 4x_1 + x_2 \\ \text{st} \quad & x_1 + 2x_2 = 6 \\ & x_1 - x_2 \geq 3 \\ & 2x_1 + x_2 \leq 10 \\ & x_1, x_2 \geq 0 \end{aligned} \quad \Rightarrow$$

x_1	x_2	e_1	s_1	a_1	a_2	rhs
0	0	0	7/3	$M-2/3$	M	58/3
0	1	0	$-1/3$	2/3	0	2/3
1	0	0	2/3	$-1/3$	0	14/3
0	0	1	1	-1	-1	1

   (a) Find the dual of this LP and its optimal solution.

   SOLUTION:

   $$\begin{aligned} \min w = \quad & 6y_1 + 3y_2 + 10y_3 \\ \text{st} \quad & y_1 + y_2 + 2y_3 \geq 4 \\ & 2y_1 - y_2 + y_3 \geq 1 \end{aligned}$$

   with y_1 URS, $y_2 \leq 0$, and $y_3 \geq 0$. The optimal solution to the dual is $\mathbf{y} = [-2/3, 0, 7/3]$.

   (b) Find the range of values of $b_3 = 10$ for which the current basis remains optimal.

   SOLUTION: To find the range of values, we have

   $$B^{-1}\mathbf{b} + \Delta B_3^{-1} = \begin{bmatrix} 2/3 \\ 14/3 \\ 1 \end{bmatrix} + \Delta \begin{bmatrix} -1/3 \\ 2/3 \\ 1 \end{bmatrix}$$

   The tricky bit here may be to determine what the third column of B^{-1} is from the optimal tableau. To determine this, figure out which variables corresponded to the three columns of the identity (from the original tableau). In this case, the column with s_1 would be it, since the only constraint that requires a slack variable is the third one.

11. Televco produces TV tubes at three plants, shown below. We have three customers, and the profits for each depend on the plant, as shown on the right.

Customer	1	2	3
Demand	80	90	100

Plant	1	2	3
Number of Tubes	50	100	50

	Cust 1	Cust 2	Cust 3
Plant 1	75	60	69
Plant 2	79	73	68
Plant 3	85	76	70

- Formulate a balanced transportation problem that can be used to maximize profits.
- Use the NW corner method to find a BFS to the problem.
- Find an optimal solution.

SOLUTION:

Important note: We set up the transportation to **minimize** the cost of the transportation rather than to maximize. The NW corner rule doesn't take costs or profits into account, so it wouldn't change, but the value of $c_{ij} - (u_i + v_j)$ does change. In the max problem, these should all be **negative** rather than positive (it is probably easier to do this rather than negating all the profits).

After using the NW corner rule and apply MODI (modified distribution), or the "u-v" method, we get the following, where each row is a plant (last row is a dummy), and each column is a customer.

	$v_1 = 75$	$v_2 = 69$	$v_3 = 69$	
$u_1 = 0$	$\boxed{75}$ 20	$\boxed{60}$ (-9)	$\boxed{69}$ 30	50
$u_2 = 4$	$\boxed{79}$ 10	$\boxed{73}$ 90	$\boxed{68}$ (-5)	100
$u_3 = 10$	$\boxed{85}$ 50	$\boxed{76}$ (-3)	$\boxed{70}$ (-9)	50
$u_4 = 1$	$\boxed{0}$ (-76)	$\boxed{0}$ (-70)	$\boxed{0}$ 70	70
Demand	80	90	100	

By incorporating any of the other variables in as BVs, we would decrease our profit (since the values in parentheses are all negative), so this represents a BFS giving the maximum profit.

12. Five workers are available to perform four jobs. The time it takes each worker to perform each job is given below. The goal is to assign workers to jobs so as to minimize the total time required. Use the Hungarian method to solve.

	Job 1	2	3	4
Worker 1	10	15	10	15
2	12	8	20	16
3	12	9	12	18
4	6	12	15	18
5	16	12	8	12

SOLUTION: You should find one optimal way of performing the assignments is:

Worker 1 does job 3. Worker 2 does job 2. Worker 3 does nothing. Worker 4 does job 1. Worker 5 does job 4.

3. A company must meet the demands shown below for a product. Demand may be backlogged at a cost of \$5 per unit per month. All demand must be met at the end of March. Thus, if 1 unit of January demand is met during March, a cost of \$5× 2=\$10 is incurred. Monthly production capacity and unit production cost during each month are shown below. A holding cost of \$20 per unit is assessed on the inventory at the end of each month.

Month	Demand	Prod Cap	Unit Prod Cost
Jan	30	35	400
Feb	30	30	420
Mar	20	35	410

Formulate a balanced transportation problem that can be used to determine how to minimize the cost (including backlogging, holding and production costs).

HINT: To set this up, think of Jan, Feb and Mar as having supplies of $35, 30 and 35$, and demands of $30, 30, 20$ (we'll need a dummy to balance. For the costs, January can supply January at a cost of \$400 per unit, January can supply Feb at a cost of \$420 per unit, and it can supply March at a cost of \$440 per unit.

SOLUTION: Here is the optimal solution:

	Jan	Feb	Mar	Dummy	
Jan	cost 400; 30	cost 420; 5	cost 440	cost 0	35
Feb	cost 425	cost 420; 10	cost 440	cost 0; 20	30
Mar	cost 420	cost 415; 15	cost 410; 20	cost 0	35
Demand	30	30	20	20	

13. Solve the following LP (HINT: It can be put into a 2×2 transportation problem).

$$\begin{aligned} \min z = \quad & 2x_1 + 3x_2 + 4x_3 + 3x_4 \\ s.t. \quad & x_1 + x_2 \leq 4 \\ & x_3 + x_4 \leq 5 \\ & x_1 + x_3 \geq 3 \\ & x_2 + x_4 \geq 6 \end{aligned}$$

(All variables ≥ 0).

SOLUTION (showing optimal using $u - v$):

	$v_1 = 2$	$v_2 = 3$	Supply
$u_1 = 0$	cost 2; 3	cost 3; 1	4
$u_2 = 2$	cost 4; (0)	cost 3; 5	5
Demand	3	6	9

14. Find the optimal solution to the balanced transportation problem below:

SOLUTION: (also with $u - v$):

	$v_1 = 4$	$v_2 = 2$	$v_3 = -2$	
$u_1 = 0$	4 10	2 5	4 (0)	15
$u_2 = 6$	12 (2)	8 5	4 10	15
	10	10	10	

15. Consider the optimal tableau for the PowerCo problem.

    (a) Find the range of values of c_{24} for which the current basis is optimal.

	$v_1 = 6$	$v_2 = 6$	$v_3 = 10$	$v_4 = 2$	
$u_1 = 0$	8 (6)	6 10	10 25	9 (7)	35
$u_2 = 3$	9 45	12 (3)	13 5	$7 + \Delta$ $(2 + \Delta)$	50
$u_3 = 3$	14 (6)	9 10	16 (3)	5 30	40
	45	20	30	30	

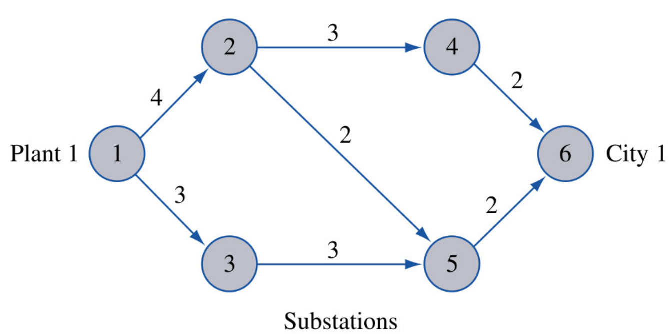
    We only need $2 + \Delta \geq 0$, or $\Delta \geq -2$ (so that $c_{24} \geq 5$).

    (b) Find the range of values of c_{23} for which the current basis is optimal.

	$v_1 = 6 - \Delta$	$v_2 = 6$	$v_3 = 10$	$v_4 = 2$	
$u_1 = 0$	8 $(2 + \Delta)$	6 10	10 25	9 (7)	35
$u_2 = 3 + \Delta$	9 45	12 $(3 - \Delta)$	$13 + \Delta$ 5	7 $(5 - \Delta)$	50
$u_3 = 3$	14 $(6 + \Delta)$	9 10	16 (3)	5 30	40
	45	20	30	30	

    Putting the inequalities together, $-2 \leq \Delta \leq 3$, or $11 < c_{23} < 16$.

16. Write the shortest path problem as (i) a transhipment problem, and (ii) a linear program. For specificity, use the PowerCo network below (Figure 2, p 414). (Hints: For transhipment, we have one supply, one demand, and a bunch of warehouses. For the LP, you could write it from the transhipment problem.). Finally, find the shortest path from Plant 1 to City 1 using Dijkstra's algorithm.



	2	3	4	5		
1	4	3	M	M	M	1
2	0	M	3	2	M	s
3	M	0	M	3	M	s
4	M	M	0	M	2	s
5	M	M	M	0	M	s
	s	s	s	s	1	1+4s

For the LP, we can also use the MCNFP framework (node constraints are $b(i) = Out - In$)

$$\begin{aligned}
\min w = \quad & 4x_{12} + 3x_{13} + 3x_{24} + 2x_{25} + 3x_{35} + 2x_{46} + 2x_{56} \\
\text{st} \quad & x_{12} + x_{13} = 1 \\
& -x_{12} + x_{24} + x_{25} = 0 \\
& -x_{13} + x_{35} = 0 \\
& -x_{24} + x_{46} = 0 \\
& -x_{25} - x_{35} + x_{56} = 0 \\
& -x_{46} - x_{56} = -1
\end{aligned}$$

with $x_{ij} \geq 0$.

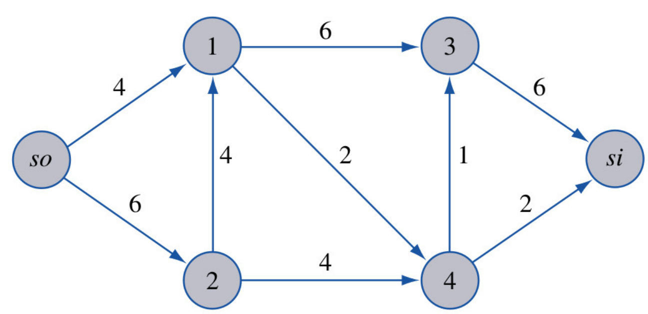
For Dijkstra's algorithm, we have:

	1	2	3	4	5	6
1	$\boxed{0_1}$	4_1	3_1	∞_1	∞_1	∞_1
3		4_1	$\boxed{3_1}$	∞_1	6_3	∞_1
2		$\boxed{4_1}$		7_2	$6_{2,3}$	∞_1
5				7_2	$\boxed{6_{2,3}}$	∞_1
4				$\boxed{7_2}$		8_5
6						$\boxed{8_5}$

This tells us that the shortest length to node 6 is 8 units, and we get that by taking either $1 \to 2 \to 5 \to 6$, or $1 \to 3 \to 5 \to 6$.

17. Given the figure below (Fig 23 from the text), first write the maximum flow problem as a linear program. (Hint: Think about the constraints on the flow for each edge, then for each vertex). Solve the max-flow problem using Ford-Fulkerson. Be sure to write out the residual graphs. Finally, find a cut giving the minimum capacity to show that your solution is correct.

FIGURE 23



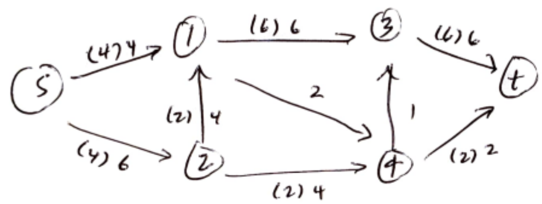
For the linear program, let v =value of flow, and make $b(so) = v$, $b(si) = -v$ (this is setting up

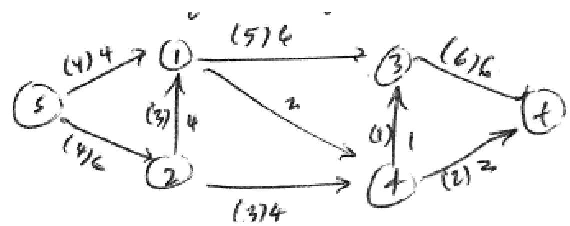
MCNFP). Then we have the following LP

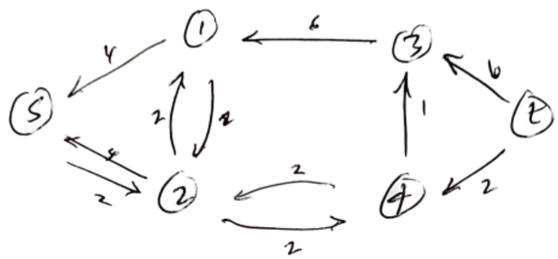
$$
\begin{array}{rrll}
\max z = & v & & \\
\text{st} & x_{s0,1} + x_{s0,2} = v & \text{Node so} & x_{so,1} \le 4 \quad x_{s0,2} \le 6 \\
& -x_{so,1} - x_{21} + x_{13} + x_{14} = 0 & \text{Node 1} & x_{13} \le 6 \quad x_{14} \le 4 \\
& -x_{so,2} + x_{21} + x_{24} = 0 & \text{Node 2} & x_{21} \le 4 \quad x_{24} \le 4 \\
& -x_{13} - x_{43} + x_{3,si} = 0 & \text{Node 3} & x_{3,si} \le 6 \\
& -x_{14} - x_{24} + x_{43} + x_{4,si} = 0 & \text{Node 4} & x_{43} \le 1 \quad x_{4,si} \le 2 \\
& -x_{4,si} - x_{6,si} = -v & \text{Node si} &
\end{array}
$$

(And all $x_{ij} \ge 0$).

There are multiple ways to get the max flow of 8. Here are two of them. The residual graph for the first is below that (and the cut can be obtained by $A = \{s0, 1, 2, 3, 4\}$, $B = \{si\}$







18. Continuing with Figure 23 from the previous question, with the maximum flow, if the cut is:

$$A = \{so, 2, 3\}, B = \{1, 4, si\}$$

then what is the net flow across the cut? What is the capacity of the cut?

SOLUTION: The net flow may be different depending on your final flow pattern. The values below will be from the first flow in the answer. List each edge as moving from A to B, or B to A, or neither. Might as well list the flow for each and capacity for each as well:

$A \to B$

Edge	f_e	c_e
$(s,1)$	4	4
$(2,1)$	2	4
$(2,4)$	2	4
$(3,t)$	6	6

$B \to A$

Edge	f_e	c_e
$(4,3)$	0	1
$(1,3)$	6	6

Neither

Edge	f_e	c_e
$(s,2)$	–	–
$(1,4)$	–	–
$(4,t)$	–	–

For the net flow, we sum f_e from A to B, subtract the flow going the other direction: $14 - 6 = 8$, and the capacity of the cut:

$$4 + 4 + 4 + 6 = 18$$