Extra Worked Example: Full Sensitivity Analysis

A factory can produce 4 products. Each product must be processed in each of two workshops. The processing times and profit margins for each of the four products is shown.

	1	2	3	4
Workshop 1	3	4	8	6
Workshop 2	6	2	5	8
Profit	4	6	10	9

If we have 400 hours of labor available in each workshop, the following LP can be used:

$$\begin{array}{lrrrrrl} \max & z = & 4x_1 & +6x_2 & +10x_3 & +9x_4 & \\ \text{st} & & 3x_1 & +4x_2 & +8x_3 & +6x_4 & \leq 400 \text{ Labor 1} \\ & & 6x_1 & +2x_2 & +5x_3 & +8x_4 & \leq 400 \text{ Labor 2} \end{array}$$

The initial and final tableaux:

x_1	x_2	x_3	x_4	s_1	s_2	
-4	-6	-10	-9	0	0	0
3	4	8	6	1	0	400
6	2	5	8	0	1	400

x_1	x_2	x_3	x_4	s_1	s_2	
$1/2$	0	2	0	$3/2$	0	600
$3/4$	1	2	$3/2$	$1/4$	0	100
$9/2$	0	1	5	$-1/2$	1	200

Sensitivity Analysis

The basic variables are (in order): $\mathcal{B} = \{x_2, s_2\}$ so that the matrices B and B^{-1} can be read directly from the initial and final tableaux respectively.

$$B = \begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 1/4 & 0 \\ -1/2 & 1 \end{bmatrix}$$

Further, the vector $\mathbf{c}^T = [4, 6, 10, 9, 0, 0]$ and the vector $\mathbf{c}_B^T = [6, 0]$.

1. Sensitivity Analysis on the NBVs.

   - x_1: Change 4 to $4 + \Delta$.
     We see that $\hat{c}_1 = 1/2$, so we have $\hat{c}_k - \Delta > 0$: $\frac{1}{2} - \Delta > 0 \quad \Rightarrow \quad \Delta < \frac{1}{2}$.
     The full computation was:
     $$-(\mathbf{c} + \Delta \mathbf{e}_1) + \mathbf{c}_B^T B^{-1} A \quad \Rightarrow \quad \hat{c} - \Delta \mathbf{e}_1$$
     so the only element of Row 0 that changes is the first element: $\frac{1}{2} - \Delta$
   - x_3: Change 10 to $10 + \Delta$. Again, $\hat{c}_3 - \Delta > 0$ gives: $2 - \Delta > 0 \quad \Rightarrow \quad \Delta < 2$.
   - x_4: Change 9 to $9 + \Delta$, and we have $\hat{c}_4 - \Delta$, or $0 - \Delta > 0$ or $\Delta < 0$.
   - We could also ask change the value of s_1 (in z). By the same reasoning of the previous variables, we would get $\frac{3}{2} - \Delta > 0$.

2. Sensitivity of BVs.

   - Change x_2 from 6 to $6 + \Delta$.
     The full computation is
     $$-(\mathbf{c} + \Delta \mathbf{e}_2) + (\mathbf{c}_B + \Delta \mathbf{e}_1)^T (B^{-1} A) = -\hat{\mathbf{c}} - \Delta \mathbf{e}_2 + \Delta \mathbf{e}_1^T (B^{-1} A)$$

That's the sum of three row vectors:

$$\begin{array}{cccccccc} & 1/2 & 0 & 2 & 0 & 3/2 & 0 \\ - & 0 & \Delta & 0 & 0 & 0 & 0 \\ + & 3\Delta/4 & \Delta & 2\Delta & 3\Delta/2 & \Delta/4 & 0 \\ \hline & \frac{1}{2}+\frac{3}{4}\Delta & 0 & 2+2\Delta & \frac{3}{2}\Delta & \frac{3}{2}+\frac{1}{4}\Delta & 0 \end{array}$$

We want all four non-zero expressions to be non-negative. Take the intersection of the four intervals, and we should see in this case that $\Delta > 0$ will satisfy all four.

- Change s_2 from 0 to Δ.
  Since s_2 is the 6th element of row 0 (and the second element of c_B), the full computation is:

$$-(\mathbf{c}+\Delta\mathbf{e}_6)+(\mathbf{c}_B+\Delta\mathbf{e}_2)^T(B^{-1}A) = -\hat{\mathbf{c}} - \Delta\mathbf{e}_6 + \Delta\mathbf{e}_2^T(B^{-1}A)$$

  This is the sum of three rows:

$$\begin{array}{ccccccc} & 1/2 & 0 & 2 & 0 & 3/2 & 0 \\ - & 0 & 0 & 0 & 0 & 0 & \Delta \\ + & 9\Delta/2 & 0 & \Delta & 5\Delta & -\Delta/2 & \Delta \\ \hline & \frac{1}{2}+\frac{9}{2}\Delta & 0 & 2+\Delta & 5\Delta & \frac{3}{2}-\frac{1}{2}\Delta & 0 \end{array} \quad \Rightarrow \quad 0 < \Delta < 3$$

3. Changes in the RHS and the Shadow Prices.

   - Change in the first constraint:
     The right hand side (RHS) of the tableau: $B^{-1}(\mathbf{b}+\Delta\mathbf{e}_1) = B^{-1}\mathbf{b} + \Delta B^{-1}\mathbf{e}_1$
     which is the old right side plus Δ times the first column of B^{-1}:

$$\left[\begin{array}{c} 100 \\ 200 \end{array}\right] + \Delta \left[\begin{array}{r} 1/4 \\ -1/2 \end{array}\right] \quad \Rightarrow \quad z = 6(100+\Delta/4) = 600 + \frac{3}{2}\Delta$$

     The shadow price for the first constraint is 3/2.
   - Change in the second constraint.
     The right hand side (RHS) of the tableau: $B^{-1}(\mathbf{b}+\Delta\mathbf{e}_2) = B^{-1}\mathbf{b} + \Delta B^{-1}\mathbf{e}_2$
     which is the old right side plus Δ times the second column of B^{-1}:

$$\left[\begin{array}{c} 100 \\ 200 \end{array}\right] + \Delta \left[\begin{array}{c} 0 \\ 1 \end{array}\right]$$

     so that $z = 600$. The shadow price is 0.
     NOTE: It makes sense that the shadow price is zero- In the optimal tableau, if $s_2 = 200$, then we have an extra 200 hours of labor available. Increasing that by 1 does nothing to z.

4. What if we introduce a new product, x_5, that has a profit of \$12.00 per unit, but is a process hog: $[8,8]^T$. Would it be worth it to bring this product in?
   SOLUTION: To price out the column, treat it as a new column of A. The new column in $B^{-1}A$ is then

$$B^{-1}\mathbf{a}_5 = \left[\begin{array}{rc} 1/4 & 0 \\ -1/2 & 1 \end{array}\right] \left[\begin{array}{c} 2 \\ 4 \end{array}\right] = \left[\begin{array}{c} 2 \\ 4 \end{array}\right]$$

   And the new Row 0 element would be

$$-12 + [6,0]\left[\begin{array}{c} 2 \\ 4 \end{array}\right]$$

   This means that bringing in Product 3 would actually create multiple optimal solutions, but would not increase the overall profit z.