A Summary of Sensitivity Analysis

We look at the kinds of changes we can make. Recall that the assumptions of this kind of analysis are: Assume that only one change at a time is made, and assume we want the current basis to remain optimal.

These are the changes we discussed, in the order that the text has them listed:

1. Change the coefficient corresponding to a **non-basic variable** (NBV).

   Since $z = \mathbf{c}_B^T B^{-1}\mathbf{b}$, then changing a non-basic variable (and keeping the current basis optimal) will not change the optimal solutions- In particular, the solution to the dual, $\mathbf{y} = (\mathbf{c}_B^T B^{-1})^T$.

   Therefore, we only need to check that the new coefficient still makes the dual solution feasible. If the NBV corresponds to the j^{th} column of A, then we only need to check the j^{th} constraint for the dual:

   $$\mathbf{y}^T\mathbf{a}_j \geq c_j$$

   You might notice that this corresponds to checking that the new Row 0 coefficient is non-negative:

   $$\mathbf{c}_B^T B^{-1}\mathbf{a}_j - c_j \geq 0$$

2. Change the coefficient corresponding to a **basic variable** (BV).

   What changes if we change a basic variable instead of a non-basic variable? Suppose we want to change the i^{th} coordinate of $\mathbf{c}_B$, that corresponds to the j^{th} column of A. Then we want to check to see if the Row 0 coefficients corresponding to NBVs are still non-negative. Before using the Row 0 formula, notice that changing coordinate i will be the same as changing $\mathbf{c}_B^T$ to:

   $$\mathbf{c}_B^T + \Delta\vec{e}_i^T$$

   where $\vec{e}_i^T$ is the i^{th} row of the $m \times m$ identity matrix (if A is $m \times n$ with rank m). Substituting this into our Row 0 formula, we get

   $$(\mathbf{c}_B^T + \Delta\vec{e}_i^T)B^{-1}A = \mathbf{c}_B^T B^{-1}A + \Delta\vec{e}_i^T B^{-1}A$$

   The important quantity here is the one attached to Δ- This is the i^{th} *row of the optimal tableau.*

   The following formula is therefore valid *for the coefficients of the non-basic variables of row 0*:

   $$\text{Old Row } 0 + \Delta \ \text{ Row } i \text{ of optimal tableau} \geq 0$$

3. Change the **right hand side** of a constraint.

   We've shown this in Exercise 9, 6.3: If we change the i^{th} RHS, then to keep the current basis optimal, we must have:

   $$B^{-1}\mathbf{b} + \Delta B^{-1}\vec{e}_i \geq 0 \quad \Rightarrow \quad \text{Old optimal RHS} + \Delta\, i^{\text{th}} \text{ col. of } B^{-1} \geq 0$$

4. Change the **column values** for a non-basic variable.

   Given the solution to the dual, $\mathbf{y}$, if we change a column corresponding to a non-basic variable, then we need to be sure that the corresponding constraint in the dual remains satisfied. In the normal case,

   $$\mathbf{y}^T\mathbf{a}_j \geq c_j$$

5. Add a new **"activity"** (or column).

   Same as the last answer; if we have a new coefficient c_j and a new column $\mathbf{a}_j$, then the current basis remains optimal if the corresponding constraint in the dual remains satisfied. Again, in the normal case, this means

   $$\mathbf{y}^T\mathbf{a}_j \geq c_j$$

6. Add a new **constraint**: Analysis is done in 6.11, but if the optimal solution satisfies the new constraint, then the optimal solution remains. If not, the current basis is no longer optimal, and we have to find the new optimal basis (using the dual simplex method).

Solutions: The Dakota Problem (p 323)

The initial and optimal tableaux are provided below.

x_1	x_2	x_3	s_1	s_2	s_3	rhs
-60	-30	-20	0	0	0	0
8	6	1	1	0	0	48
4	2	1.5	0	1	0	20
2	1.5	0.5	0	0	1	8

x_1	x_2	x_3	s_1	s_2	s_3	rhs
0	5	0	0	10	10	250
0	-2	0	1	2	-8	24
0	-2	1	0	2	-4	8
1	1.25	0	0	-0.5	1.5	2

1. Find the allowable changes in x_1, x_3 that keep the current basis optimal.

   SOLUTION: First, note that $\mathcal{B} = \{s_1, x_3, x_1\}$, so $\mathbf{c}_B^T = [0, 20, 60]$.

   (a) Changing x_1 means changing the third component of $\mathbf{c}_B$, and we'll use the third row of $B^{-1}A$. The formula below only works for the non-basic variables (the others are crossed out). Consider Row 0 $+\Delta \cdot$ Row 3 of $B^{-1}A$

   $$[\cancel{0}, 5, \cancel{0}, \cancel{0}, 10, 10] + \Delta[\cancel{1}, 5/4, \cancel{0}, \cancel{0}, -1/2, 3/2]$$

   Then the three inequalities must hold:

   $$5 + \frac{5}{4} \geq 0, \qquad 10 - \frac{1}{2}\Delta \geq 0 \quad 10 + \frac{3}{2}\Delta \geq 0 \quad \Rightarrow \quad -4 \leq \Delta \leq 20$$

   (b) Changing x_3 means changing the second component of $\mathbf{c}_B^T$, so we use the second row of $B^{-1}A$:

   $$[\cancel{0}, 5, \cancel{0}, \cancel{0}, 10, 10] + \Delta[\cancel{0}, -2, \cancel{1}, \cancel{0}, 2, -4]$$

   Which gives us:

   $$5 - 2\Delta \geq 0 \qquad 10 + 2\Delta \geq 0 \qquad 10 - 4\Delta \geq 0 \quad \Rightarrow \quad -5 \leq \Delta \leq \frac{5}{2}$$

2. Find the allowable change in x_2 that keeps the current basis optimal:

   SOLUTION: This is a NBV, so we only need to track $\mathbf{y}^T\mathbf{a}_2 - c_2 \geq 0$. In this case,

   $$[0, 10, 10] \cdot [6, 2, 3/2]^T - (30 + \Delta) = 20 + 15 - (30 + \Delta) = 5 - \Delta \geq 0 \quad \Rightarrow \quad \Delta \leq 5$$

3. Give the allowable changes to the RHS:

   (a) For b_1, we have:

   $$B^{-1}\mathbf{b} + \Delta(B^{-1})_1 = \begin{bmatrix} 24 \\ 8 \\ 2 \end{bmatrix} + \Delta \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \geq 0 \quad \Rightarrow \quad -24 \leq \Delta$$

   (b) For b_2, we have:

   $$B^{-1}\mathbf{b} + \Delta(B^{-1})_2 = \begin{bmatrix} 24 \\ 8 \\ 2 \end{bmatrix} + \Delta \begin{bmatrix} 2 \\ 2 \\ -1/2 \end{bmatrix} \geq 0 \quad \Rightarrow \quad -4 \leq \Delta \leq 4$$

   (c) For b_3, we have:

   $$B^{-1}\mathbf{b} + \Delta(B^{-1})_2 = \begin{bmatrix} 24 \\ 8 \\ 2 \end{bmatrix} + \Delta \begin{bmatrix} -8 \\ -4 \\ -3/2 \end{bmatrix} \geq 0 \quad \Rightarrow \quad -\frac{4}{3} \leq \Delta \leq 2$$

4. What happens if a table sells for \$43, and uses 5 ft of lumber, 2 finishing hours, and 2 carpentry hours.

   SOLUTION: This changes the second column of A. We want to check to see if $\mathbf{y}^T\mathbf{a}_2 \geq c_2$. In this case,

   $$0(5) + 10(2) + 10(2) - 43 = -3 \leq 0$$

   Therefore, this will change the optimal basis (in fact, this makes x_2 something that will be constructed).

5. New activity: Suppose we consider making footstools, at $15 selling price, and uses 1 ft of lumber, 1 finishing hour and 1 carpentry hour. Should we make footstools?

   SOLUTION: This new column for the primal becomes a constraint in the dual. We check that the current dual remains feasible:

$$0(1) + 10(1) + 10(1) - 15 = 5$$

   Yes, the current dual remains feasible, so the current optimal basis does not change. We should NOT make footstools.

6. For adding a new constraint to the primal, see the Dual Simplex Method.