Review Solutions, Exam 2, Operations Research

1. Prove the weak duality theorem: For any $\mathbf{x}$ feasible for the primal and $\mathbf{y}$ feasible for the dual, then...

   HINT: Put the primal so that $A\mathbf{x} \le \mathbf{b}$ and the dual so that $A^T\mathbf{y} \ge \mathbf{c}$

   SOLUTION: With the primal and dual in normal form, then

   $$\mathbf{y}^T A\mathbf{x} \le \mathbf{y}^T\mathbf{b} \qquad \text{and} \qquad \mathbf{x}^T A^T\mathbf{y} \ge \mathbf{x}^T\mathbf{c}$$

   Noting that $\mathbf{x}^T A^T\mathbf{y} = (A\mathbf{x})^T\mathbf{y} = \mathbf{y}^T(A\mathbf{x})$, we get that:

   $$\mathbf{c}^T\mathbf{x} \le \mathbf{y}^T A\mathbf{x} \le \mathbf{b}^T\mathbf{y}$$

2. Show that the solution to the dual is $\mathbf{y} = (\mathbf{c}_B^T B^{-1})^T$ (if the primal and dual are both feasible).

   HINT: Strong duality might be useful.

   SOLUTION: Ignore the hint, which should have read: "Consider Row 0".

   In the optimal tableau, the coefficients of Row 0 are all non-negative. Therefore,

   $$\mathbf{c}_B^T B^{-1}A - \mathbf{c}^T \ge 0 \quad \Rightarrow \quad \mathbf{c}_B^T B^{-1}A \ge \mathbf{c}^T$$

   The vectors on the right and left are in rows. Transpose them for columns:

   $$A^T(\mathbf{c}_B^T B^{-1})^T \ge \mathbf{c}$$

   Therefore, if we define $\mathbf{y} = (\mathbf{c}_B^T B^{-1})^T$, we know that $\mathbf{y}$ is feasible for the dual.

   Now, is $\mathbf{y}$ optimal for the dual? Now we can use strong duality:

   $$\mathbf{y}^T\mathbf{b} = \mathbf{c}_B^T B^{-1}\mathbf{b} = \mathbf{c}^T\mathbf{x}$$

   so $\mathbf{y}$ is optimal for the dual.

3. Solve using big-M:

   $$\begin{array}{rl} \max \quad z = & 2x_1 + 3x_2 \\ \text{st} & 2x_1 + x_2 \ge 4 \\ & x_1 - x_2 \ge -1 \\ & \qquad x_2 \le 3 \\ & x_1, x_2 \ge 0 \end{array}$$

   Side remark: In the third constraint, I meant to make the first variable less than 3, but we'll go ahead and solve the one that is written.

   SOLUTION: Notice that we do not want the negative b_2, so multiply that constraint by -1. This makes constraints 2 and 3 "normal" (with slack variables), so we only need one excess variable for the first constraint. This means we have only one artificial variable, and the tableau is:

x_1	x_2	e_1	s_2	s_3	a_1	rhs
-2	-3	0	0	0	M	0
2	1	-1	0	0	1	4
-1	1	0	1	0	0	1
0	1	0	0	1	0	3

$\rightarrow$

x_1	x_2	e_1	s_2	s_3	a_1	rhs
$-2-2M$	$-3-M$	M	0	0	0	$-4M$
2	1	-1	0	0	1	4
-1	1	0	1	0	0	1
0	1	0	0	1	0	3

Now, after pivoting in Column 1, and second row (counting the top as the first), we get:

x_1	x_2	e_1	s_2	s_3	a_1	rhs
0	-2	-1	0	0	$1+M$	4
1	1/2	$-1/2$	0	0	1/2	2
0	3/2	$-1/2$	1	0	1/2	3
0	1	0	0	1	0	3

At this stage, we might ignore the artificial variable, and the rest of the problem is the standard simplex method. The optimal tableau is:

x_1	x_2	e_1	s_2	s_3	a_1	rhs
0	0	0	-2	5	M	13
1	0	0	-1	1	0	2
0	1	0	0	1	0	3
0	0	1	-2	3	-1	3

And we see that the feasible set (and the LP) is unbounded.

4. Solve the last problem again using the dual simplex method.

   NOTE: We end up back at the regular simplex method after the first step.

x_1	x_2	e_1	s_2	s_3	rhs		x_1	x_2	e_1	s_2	s_3	rhs
-2	-3	0	0	0	0		0	-2	-1	0	0	4
-2	-1	1	0	0	-4	$\rightarrow$	1	1/2	$-1/2$	0	0	2
-1	1	0	1	0	1		0	3/2	$-1/2$	1	0	3
0	1	0	0	1	3		0	1	0	0	1	3

   (You can check that $(2, 0)$ satisfies the constraints). If we continue with the usual simplex method, we get the same tableau as before (so the LP is unbounded).

5. Going back to the original LP in (1), if the constraint $x_1 + x_2 \leq 4$ is added, does the basis stay optimal? If not, find the new basis (and the new solution).

   SOLUTION: Graphically, this makes the feasible set bounded. If we put in $x_1 = 2, x_2 = 3$, the constraint is not satisfied, which means that this solution is not the optimal one with the new constraint:

x_1	x_2	e_1	s_2	s_3	s_4	rhs
0	0	0	-2	5	0	13
1	0	0	-1	1	0	2
0	1	0	0	1	0	3
0	0	1	-2	3	0	3
1	1	0	0	0	1	4

(1)

To initialize the array, we'll pivot on columns 1 and 2:

x_1	x_2	e_1	s_2	s_3	s_4	rhs
0	0	0	-2	5	0	13
1	0	0	-1	1	0	2
0	1	0	0	1	0	3
0	0	1	-2	3	0	3
0	0	0	1	-2	1	-1

(2)

And now we see that we can pivot in the fourth column (which gives us the positive Row 0 to start the dual simplex):

x_1	x_2	e_1	s_2	s_3	s_4	rhs
0	0	0	0	1	2	11
1	0	0	0	1	1	1
0	1	0	0	1	0	3
0	0	1	0	-1	2	1
0	0	0	1	-2	1	-1

(3)

Now we use the dual simplex algorithm, and pivot in the fifth column to try to get something primal-feasible.

x_1	x_2	e_1	s_2	s_3	s_4	rhs
0	0	0	$1/2$	0	$5/2$	$21/2$
1	0	0	$-1/2$	0	$1/2$	$3/2$
0	1	0	$1/2$	0	$1/2$	$5/2$
0	0	1	$-1/2$	0	$3/2$	$3/2$
0	0	0	$-1/2$	1	$-1/2$	$1/2$

(4)

EXTRA: See if you understood what just happened in terms of the primal and dual. For your convenience, the constraints are:

$$\begin{aligned} 2x_1 + x_2 &\geq 4 \\ -x_1 + x_2 &\leq 1 \\ x_2 &\leq 3 \\ x_1 + x_2 &\leq 4 \end{aligned} \qquad\qquad \begin{aligned} 2y_1 - y_2 + y_4 &\geq 2 \\ y_1 + y_2 + y_3 + y_4 &\geq 3 \end{aligned}$$

Now, in Equation (1), the original solution, $x_1 = 2, x_2 = 3$ satisfied the first three constraints. We incorporate the last constraint to get Equation (2).

The negative sign in the RHS indicates that $(2, 3)$ is no longer feasible in the primal. The negative sign in Row 0 is not feasible for the dual (since $y_2 \geq 0$). However, pivoting in the fourth column to get Equation (3), we now have a solution that is feasible for the dual: $y_1 = y_2 = 0$, $y_3 = 1, y_4 = 2$. The value you see in the upper right corner is $\mathbf{y}^T\mathbf{b} = 11$. However, we still do not have a feasible point for the primal.

Pivoting in the last step to get Equation (4) gets us to a point that is feasible in the primal: $x_1 = 3/2$, $x_2 = 5/2$ AND the dual: $y_1 = y_3 = 0$, $y_2 = 1/2$ and $y_4 = 5/2$, and we now have the optimal solution for both the primal and dual!

6. Using the big-M method on a maximization problem, I got the following tableau:

	x_1	x_2	x_3	s_1	e_1	e_2	a_1	a_2	rhs
	$-1/2+2M$	$-5/2+M$	M	$1/2+M$	M	M	0	0	$2-3M$
x_3	1/2	1/2	1	1/2	0	0	0	0	2
a_1	−3/2	−1/2	0	−1/2	−1	0	1	0	2
a_2	−1/2	−1/2	0	−1/2	0	−1	0	1	1

Should I stop or should I go? If I stop, what should I conclude?

SOLUTION: Please stop! The conclusion is that the LP is infeasible.

7. Use the two phase method to solve the following

$$\begin{array}{rl} \max \quad z = & x_1 + x_2 \\ \text{st} & x_1 - x_2 - x_3 = 1 \\ & -x_1 + x_2 + 2x_3 - x_4 = 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

SOLUTION: Starting things off, recall that we change our objective function to: max $z = -a_1 - a_2$ or $z + a_1 + a_2 = 0$. Incorporating this into the tableau, we get the following. We initialize by making the last two columns pivot columns as shown:

x_1	x_2	x_3	x_4	a_1	a_2	rhs
0	0	0	0	1	1	0
1	−1	−1	0	1	0	1
−1	1	2	−1	0	1	1

$\rightarrow$

x_1	x_2	x_3	x_4	a_1	a_2	rhs
0	0	−1	1	0	0	−2
1	−1	−1	0	1	0	1
−1	1	2	−1	0	1	1

Now we perform the Simplex Method, and we get:

x_1	x_2	x_3	x_4	a_1	a_2	rhs
0	0	0	0	1	1	0
1	−1	0	−1	2	1	3
0	0	1	−1	1	1	2

$\rightarrow$

x_1	x_2	x_3	x_4	a_1	a_2	rhs
0	0	0	0	1	1	0
1	−1	0	−1	2	1	3
0	0	1	−1	1	1	2

Now we have a feasible point. Remove the artificial variables and put in the original Row 0 coefficients, then make columns 1 and 3 pivot columns again. We then get:

x_1	x_2	x_3	x_4	rhs
0	−2	0	−1	3
1	−1	0	−1	3
0	0	1	−1	2

From which we conclude that we have an unbounded LP.

8. Consider the LP and the optimal tableau with missing Row 0.

$$\begin{array}{rl} \max z = & 3x_1 + x_2 \\ \text{s.t.} & 2x_1 + x_2 \leq 4 \\ & 3x_1 + 2x_2 \geq 6 \\ & 4x_1 + 2x_2 = 7 \\ & x_1, x_2 \geq 0 \end{array}$$

x_1	x_2	s_1	e_2	a_2	a_3	rhs
0	0	1	0	0	$-\frac{1}{2}$	$\frac{1}{2}$
0	1	0	−2	2	$-\frac{3}{2}$	$\frac{3}{2}$
1	0	0	1	−1	1	1

Find Row 0.

SOLUTION: Row 0 coefficients are given by

$$\mathbf{c}_B^T B^{-1} A - \mathbf{c} \qquad \text{where} \qquad \mathbf{c}_B^T = [0, 1, 3]$$

However, we can compute them individually. The first three are zero.

The coefficient for e_2: $[0, 1, 3][0, -2, 1]^T = 1$

The coefficient for a_2: $[0, 1, 3][0, 2, -1] + M = 1 + M$

The coefficient for a_3: $[0, 1, 3][-1/2, -3/2, 1]^T + M = 3/2 + M$

The optimal RHS is: $[0, 1, 3][1/2, 3/2, 1]^T = 9/2$

Therefore, the optimal tableau is:

x_1	x_2	s_1	e_2	a_2	a_3	rhs
0	0	0	1	$-1 + M$	$3/2 + M$	$9/2$
0	0	1	0	0	$-\frac{1}{2}$	$\frac{1}{2}$
0	1	0	-2	2	$-\frac{3}{2}$	$\frac{3}{2}$
1	0	0	1	-1	1	1

9. Consider the following LP and its optimal tableau:

$$\begin{aligned} \max \quad & z = 4x_1 + x_2 + 2x_3 \\ \text{st} \quad & 8x_1 + 3x_2 + x_3 \le 12 \\ & 6x_1 + x_2 + x_3 \le 8 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

x_1	x_2	x_3	s_1	s_2	rhs
8	1	0	0	2	16
2	2	0	1	-1	4
6	1	1	0	1	8

(a) Find the range of values for the coefficient of x_3 which keeps the current basis optimal.

SOLUTION: We see that x_3 is a basic variable, so we change its value by substituting $2 + \Delta$ in for 2 in the vector $\mathbf{c}_B^T$, then we check the effect of that on Row 0 for non-basic variables. we showed in class that the effect of that is to take the old Row 0, then we add Δ times the second row of the optimal table- Recall we only check non-basic variables:

$$[8, 1, 0, 0, 2] + \Delta[6, 1, 0, 0, 1] \ge 0 \quad \Rightarrow \quad \begin{array}{rl} 8 + 6\Delta & \ge 0 \\ 1 + \Delta & \ge 0 \\ 2 + \Delta & \ge 0 \end{array} \quad \Rightarrow \quad \Delta \ge -1$$

(b) Find the range of values for the coefficient of x_1 which keeps the current basis optimal.

SOLUTION: We check $\mathbf{c}_B^T B^{-1} \mathbf{a}_1 - (c_1 + \Delta) \ge 0$, or in this case,

$$[0, 2] \cdot [2, 6]^T - (4 + \Delta) \ge 0 \quad \Rightarrow \quad \Delta \le 8$$

(c) Find the range of values for the RHS of each constraint that keeps the current basis optimal.

SOLUTION: For b_1, we have:

$$B^{-1}\mathbf{b} + \Delta(B^{-1})_1 = \left[\begin{array}{c} 4 \\ 8 \end{array}\right] + \Delta \left[\begin{array}{c} 1 \\ 0 \end{array}\right] \geq 0 \quad \Rightarrow \quad -4 \leq \Delta$$

For b_2, we have:

$$B^{-1}\mathbf{b} + \Delta(B^{-1})_2 = \left[\begin{array}{c} 4 \\ 8 \end{array}\right] + \Delta \left[\begin{array}{r} -1 \\ 1 \end{array}\right] \geq 0 \quad \Rightarrow \quad -8 \leq \Delta \leq 4$$

(d) Write the dual, and solve it using the tableau for the primal (given above).
SOLUTION: The dual is in normal form, so we can write it directly:

$$\begin{array}{rrl} \min w = & 12y_1 + 8y_2 & \\ \text{st} & 8y_2 + 6y_2 & \geq 4 \\ & 3y_1 + y_2 & \geq 1 \\ & y_2 + y_2 & \geq 2 \\ & y_1, y_2 & \geq 0 \end{array} \quad \Rightarrow \quad \mathbf{y} = \left[\begin{array}{c} 0 \\ 2 \end{array}\right]$$

10. Give an argument why, if the primal is unbounded, then the dual must be infeasible.

SOLUTION:

Suppose the dual was feasible. Then there exists a $\mathbf{y}$ that satisfies the constraints for the dual. But in that case, for every $\mathbf{x}$ in the primal,

$$\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$$

But we said that the primal was unbounded, so that $\mathbf{c}^T\mathbf{x} \to \infty$. This is a contradiction- Thus, the dual must be infeasible.

Chapter 6 Review Problems:

3, 4 (except 4(b)), 6, 9, 10, 16, 17, 20, 21, 23, 33, 34.