Definitions and Theorems: Chapter 4

- A **basic solution** to $A\mathbf{x} = \mathbf{b}$, where A is $m \times n$ with rank m, is found by choosing m linearly independent columns of A and setting the remaining $n - m$ variables to zero.

- Vars corresponding to the linearly indep cols are **basic variables** (BV).

- The other vars are the **nonbasic variables** (NBV).

- If one or more of the basic variables is zero (also), then the basic solution is called **degenerate**, otherwise it is **nondegenerate**.

- The basic solution is **feasible** if every element of the vector is non-negative. In this case, the solution is called a **basic feasible solution**, or BFS.

- Given a convex set S, point $\mathbf{x}$ is an **extreme point** if each line segment that lies completely in S and contains the point $\mathbf{x}$ has $\mathbf{x}$ as an endpoint.

  Alternatively, given a line segment containing $\mathbf{x}$ that lies entirely in S so that:

  $$\mathbf{x} = \lambda\mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2$$

  for some $\mathbf{x}_1, \mathbf{x}_2$ in S, then $\lambda = 0$ or $\lambda = 1$.

  As a second alternative definition, $\mathbf{x}$ is an extreme point of S if there is no way that $\mathbf{x}$ can be expressed as $\lambda\mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2$ for $0 < \lambda < 1$ (and $\mathbf{x}_1, \mathbf{x}_2 \in S$).

- A point $\mathbf{x}$ is a vertex of the polyhedron P if there is a vector $\mathbf{c}$ such that

  $$\mathbf{c}^T\mathbf{x} < \mathbf{c}^T\mathbf{y}$$

  for all $\mathbf{y} \neq \mathbf{x} \in P$. This means that there is an (affine) hyperplane for which P is entirely contained on one side, and the only point of intersection is $\mathbf{x}$.

  NOTE: In this case, we are thinking of the plane as being defined by $\mathbf{c}^T\mathbf{x} + b = 0$.

- **Definition:** Let A be an $m \times n$ matrix with rank m, and let $S = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{b}\}$ be the feasible region of some LP in standard form. A vector $\mathbf{d} \neq 0$ is called a **direction of unboundedness** (for either S or the LP) if, for every $\mathbf{x}$ and number $\lambda \geq 0$, the ray $\mathbf{x} + \lambda\mathbf{d} \in S$

- Lemma 1 (to be proven): For an LP in standard form $A\mathbf{x} = \mathbf{b}$, and $\mathbf{x} \geq 0$, the vector $\mathbf{d}$ is a direction of unboundedness iff $A\mathbf{d} = 0$ and $\mathbf{d} \geq 0$ (therefore, $\mathbf{d}$ is in the null space of A)

- Lemma 2 (to be proven): For an LP in standard form, where we optimize $\mathbf{c}^T\mathbf{x}$ and if $\mathbf{d}$ is a direction of unboundedness, then $\mathbf{c}$ and $\mathbf{d}$ are orthogonal.

- *Definition:* Let $\mathbf{x}_1, \ldots, \mathbf{x}_k$ be points in $\mathbb{R}^n$. Then we'll recall that a **linear combination** is any sum of the form:

  $$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k$$

  for any scalars $c_1, \ldots, c_k$.

- *Definition:* A **convex combination** of the (vectors or points) is any sum of the form

$$\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \cdots + \lambda_k\mathbf{x}_k$$

  where $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$.

- *Definition:* The **convex hull** of $\mathbf{x}_1, \ldots, \mathbf{x}_k$ is the set of all convex combinations of our k points.

- **Lemma 3:** Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$, and let $\sigma_i \geq 0$ and $\sum \sigma_i = 1$. Then

$$\lambda_1 \leq \sigma_1\lambda_1 + \sigma_2\lambda_2 + \cdots + \sigma_n\lambda_n \leq \lambda_n$$

  "Any convex combination of numbers will be larger than the minimum of those numbers and smaller than the maximum."

- **The Representation Theorem (Text's Thm 2)** Consider an LP in standard form with a non-empty feasible region $S = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$, with k BFS (or vertices, to be proven), $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_k$.

  Then any point in S can be written as a combination of $\mathbf{d}$ and a **convex combination** of the extreme points. In other words, we can express $\mathbf{x}$ as:

$$\mathbf{x} = c\mathbf{d} + \sum_{i=1}^{k} \sigma_i \mathbf{b}_i$$

  with $\sigma_i \geq 0$ and $\sum \sigma_i = 1$ and $c > 0$.

- **Theorem 3: The Fundamental Theorem of Linear Programming**

  > If there is a solution to a linear programming problem, then it will occur at an extreme point, or on a line segment between two corner points.