A Fundamental Theorem

There is something called *The Fundamental Theorem of Linear Programming*, which goes something like this:

> If there is a solution to a linear programming problem, then it will occur at an extreme point, or on a line segment between two corner points. (This does not preclude the case of more than two corner points)

Our text uses the following theorem (**Theorem 3**) instead: If an LP has an optimal solution, then it has an optimal BFS (and by Theorem 1, it has an optimal extreme point).

Before we get into the proof, consider the following Lemma:

> Let $\mathbf{x}$ be in the feasible set of a linear program. Then by the Representation Theorem, we can write:
>
> $$\mathbf{x} = \mathbf{d} + \sum_{i=1}^{k} \sigma_i \mathbf{b}_i$$
>
> (with the caveats from the Representation Theorem). We notice that from this representation, I can multiply the vector $\mathbf{d}$ by any non-negative constant λ and remain in the feasible set:
>
> $$\mathbf{x}' = \lambda \mathbf{d} + \sum_{i=1}^{k} \sigma_i \mathbf{b}_i$$
>
> **Exercise 1:** Prove this statement.

Proof:

Let $\mathbf{x}$ be optimal. We want to show $\mathbf{x}$ is a basic feasible solution (or equivalently, an extreme point or vertex).

Since $\mathbf{x}$ is feasible, by the Representation Theorem we can write as a combination of a direction of unboundedness and the vertices of the feasible set:

$$\mathbf{x} = \mathbf{d} + \sum_{i=1}^{k} \sigma_i \mathbf{b}_i$$

Saying that this is an optimal solution implies that $\mathbf{c}^T\mathbf{x}$ is a maximum. Therefore, the following is a maximum (over all possible $\mathbf{x}$):

$$\mathbf{c}^T\mathbf{x} = \mathbf{c}^T\mathbf{d} + \sum_{i=1}^{k} \sigma_i \mathbf{c}^T \mathbf{b}_i$$

We now show that because this is optimal, the direction of unboundedness must be orthogonal to $\mathbf{c}$. Why is this? Suppose not:

- Case 1: $\mathbf{d} = \vec{0}$. In that case, $\mathbf{x}$ is a convex combination of BFS (go to the next step of the proof).

- Case 2: $\mathbf{c}^T\mathbf{d} > 0$. If this is true, let

$$\mathbf{x}' = 10\mathbf{d} + \sum_{i=1}^{k} \sigma_i \mathbf{b}_i$$

  where the σ_i are from the representation for $\mathbf{x}$. Then $\mathbf{c}^T\mathbf{x}'$ is larger than $\mathbf{c}^T\mathbf{x}$. But this is a contradiction, as this was a maximum.

- Case 3: $\mathbf{c}^T\mathbf{d} < 0$. If this is the case, then (set $\mathbf{d} = 0$) and consider $\mathbf{c}^T\mathbf{x}'$ again, where:

$$\mathbf{x}' = \sum_{i=1}^{k} \sigma_i \mathbf{b}_i$$

  This is again larger than $\mathbf{c}^T\mathbf{x}$ (which is again a contradiction).

Therefore, if $\mathbf{c}^T\mathbf{x}$ is a maximum, then $\mathbf{c}$ and $\mathbf{d}$ are orthogonal (or $\mathbf{d} = \vec{0}$).

The next part of the proof relies on the following "Lemma":

Lemma: Let $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$, and let $\sigma_1, \ldots, \sigma_n$ be non-negative so that $\sum \sigma_i = 1$. Then show that

$$\lambda_1 \le \sigma_1\lambda_1 + \sigma_2\lambda_2 + \cdots + \sigma_n\lambda_n \le \lambda_n$$

Proof (just one way):

$$\sigma_1\lambda_1 + \sigma_2\lambda_2 + \cdots + \sigma_n\lambda_n \le \sigma_1\lambda_n + \sigma_2\lambda_n + \cdots + \sigma_n\lambda_n = \lambda_n$$

Now, consider

$$\mathbf{c}^T\mathbf{x} = \mathbf{c}^T\mathbf{d} + \sum_{i=1}^{n} \sigma_i \mathbf{c}^T\mathbf{b}_i = 0 + \sigma_1\mathbf{c}^T\mathbf{b}_1 + \cdots \sigma_n\mathbf{c}^T\mathbf{b}_n$$

From what we just showed,

$$\min_i \left\{\mathbf{c}^T\mathbf{b}_i\right\} \le \mathbf{c}^T\mathbf{x} \le \max_i \left\{\mathbf{c}^T\mathbf{b}_i\right\}$$

Therefore, the optimal value occurs at a BFS.

Definition: Adjacent Solutions

Def: For an LP in standard form (A is $m \times n$ with rank m), two BFS are said to be **adjacent** is they share $m-1$ basic variables (only 1 basic variable is different).

For example, BFS using x_1, x_2, x_3 as the BV would be adjacent to one using x_1, x_2, x_4 but not adjacent to a BFS using BV's x_1, x_4, x_5.

Example

Consider the LP:

$$\begin{array}{rlll} \min z = & x_1 + & x_2 & \\ \text{s.t.} & x_1 & +x_2 \geq 250 & \\ & 2x_1 & +x_2 \geq 400 & \\ & x_1, & x_2 \geq 0 & \end{array}$$

We want to illustrate the points from the proof of the Fundamental Theorem. It is easy to show that this problem is maximized at $(150, 100)$.

1. Write the problem in standard form and give the vector $\mathbf{c}$.

   SOLUTION: Standard form would be given as $\mathbf{c}^T\mathbf{x}$, such that $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 2 & 1 & 0 & -1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ e_1 \\ e_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 250 \\ 400 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

2. Give a description of the directions of unboundedness.

   SOLUTION: In the (x_1, x_2) plane, any vector pointing to the right with any angle between 0 and $\pi/2$ will work. Given x_1 and x_2, the other two dimensions can be computed since the vector is in the null space of A:

$$\begin{aligned} e_1 &= x_1 + x_2 \\ e_2 &= 2x_1 + x_2 \end{aligned}$$

   For example, if we take the direction $[1, 1]^T$ in the (x_1, x_2) plane, the vector $\mathbf{d} = [1, 1, 2, 3]^T$ (and this is in the null space of A).

3. In two dimensions, the problem is minimized if $x_1 = 150, x_2 = 100$. Find the corresponding 4-dimensional vector $\mathbf{x}$.

   SOLUTION: The solution is the intersection between the two lines, so $e_1 = 0$ and $e_2 = 0$, and $\mathbf{x} = [150, 100, 0, 0]$

4. Continuing with the last question, is $\mathbf{c}^T\mathbf{d} = 0$? Is that a problem?