Example 1, Section 4.16

The Leon Burnit Ad Agency is trying to determine a TV schedule for Priceler Auto.

Ad	HIM	LIP	HIW	Cost
Football	7	10	5	100,000
Soap Opera	3	5	4	60,000
Goals	40	60	35	600,000

This means, for example, that the ad agency wants to reach at least 40 million viewers "HIM", 60 million "HIW", and 35 million "LIP", with a total budget of $600,000.00.

SOLUTION (for the penalty terms):

$$\begin{array}{lrll}
\min z = & 200s_1^- + 100s_2^- + 50s_3^- + s_4^+ & & \\
\text{s.t.} & 7x_1 + 3x_2 + s_1^- - s_1^+ & = 40 & \text{HIM} \\
 & 10x_1 + 5x_2 + s_2^- - s_2^+ & = 60 & \text{LIP} \\
 & 5x_1 + 4x_2 + s_3^- - s_3^+ & = 35 & \text{HIW} \\
 & 100x_1 + 60x_2 & \leq 600 & \text{Budget}
\end{array}$$

The initial tableau:

$$\begin{array}{rrrrrrrrrr|r}
x_1 & x_2 & s_1^- & s_1^+ & s_2^- & s_2^+ & s_3^- & s_3^+ & s_4^- & s_4^+ & \text{rhs} \\
0 & 0 & 200 & 0 & 100 & 0 & 50 & 0 & 0 & 1 & 0 \\
\hline
7 & 3 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 40 \\
10 & 5 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 60 \\
5 & 4 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 35 \\
100 & 60 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 600
\end{array}$$

Final tableau:

$$\left[\begin{array}{cccccccccc|c}
x_1 & x_2 & s_1^- & s_1^+ & s_2^- & s_2^+ & s_3^- & s_3^+ & s_4^- & s_4^+ & \text{rhs} \\
0 & 0 & 200 & 0 & \frac{280}{3} & \frac{20}{3} & \frac{130}{3} & \frac{20}{3} & 1 & 0 & -\frac{100}{3} \\
\hline
1 & 0 & 0 & 0 & \frac{4}{15} & -\frac{4}{15} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{13}{3} \\
0 & 0 & -1 & 1 & \frac{13}{15} & -\frac{13}{15} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\
0 & 0 & 0 & 0 & \frac{20}{3} & -\frac{20}{3} & \frac{20}{3} & -\frac{20}{3} & -1 & 1 & \frac{100}{3} \\
0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} & 0 & 0 & \frac{10}{3}
\end{array}\right]$$

Conclusion:

$$x_1 = \frac{13}{3} \qquad x_2 = \frac{10}{3}$$

Which gives:

$$s_1^- = 0, s_1^+ = \frac{1}{3} \qquad s_2^- = 0, s_2^+ = 0 \qquad s_3^- = 0, s_3^+ = 0 \qquad s_4^- = 0, s_4^+ = 33\frac{1}{3}$$

CONCLUSION: An increase in the budget by $33\frac{1}{3}$ thousands of dollars would allow us to meet all of our advertising goals.

Example 2, Section 4.16 (Goal Programming)

The Dewright company is considering three new products to replace current models. ...

Factor	Prod 1	2	3	Goal	(units)	Penalty
Long run profit	12	9	15	≥ 125	(millions dollars)	5
Employment	5	3	4	$= 40$	(hundreds of people)	$2(+), 4(-)$
Capital	5	7	8	≤ 55	(millions dollars)	3

SOLUTION: As before, let s_i^+ be the amount over, s_i^- be the amount under the stated goal. In goal 1, we see that the amount under gets the penalty (s_1^-), in employment, we get both, and in capital, the penalty is for going over (s_3^+).

$$\min \quad z = 5s_1^- + 2s_2^+ + 4s_2^- + 3s_3^+$$

$$\begin{aligned} 12x_1 + 9x_2 + 15x_3 + s_1^- - s_1^+ &= 125 \\ 5x_1 + 3x_2 + 4x_3 + s_2^- - s_2^+ &= 40 \\ 5x_1 + 7x_2 + 8x_3 + s_3^- - s_3^+ &= 55 \end{aligned}$$

Solution in LINDO:

```
min 5s1m+2s2p+4s2m+3s3p
st  12x1+9x2+15x3+s1m-s1p=125
     5x1+3x2+ 4x3+s2m-s2p=40
     5x1+7x2+ 8x3+s3m-s3p=55

LINDO OUTPUT:

LP OPTIMUM FOUND AT STEP      5

        OBJECTIVE FUNCTION VALUE

        1)      16.66667

  VARIABLE        VALUE          REDUCED COST
       S1M         0.000000          3.095238
       S2P         8.333333          0.000000
       S2M         0.000000          6.000000
       S3P         0.000000          0.428571
        X1         8.333333          0.000000
        X2         0.000000          6.857143
        X3         1.666667          0.000000
       S1P         0.000000          1.904762
       S3M         0.000000          2.571429
```

We see Goal 1 was attained, we went over Goal 2 by 25/3, and met Goal 3 by taking $x_1 = 25/3$, $x_2 = 0$ and $x_3 = 5/3$.

Example 3: Preemptive Goal Programming, In-Class

Going back to Example 1, suppose we prioritize the goals:

$$\begin{array}{lll} \min & z = P_1 s_1^- & & \\ \min & z = & P_2 s_2^- & \\ \min & z = & & P_3 s_3^- \end{array}$$

To reiterate, the "constraints" were:

$$\begin{array}{rll} 7x_1 + 3x_2 & \geq 40 & \text{HIM} \\ 10x_1 + 5x_2 & \geq 60 & \text{LIP} \\ 5x_1 + 4x_2 & \geq 35 & \text{HIW} \\ 100x_1 + 60x_2 & \leq 600 & \text{Budget} \end{array}$$

Write the tableau that we will put into Maple:

$$\begin{array}{rr|rrr|rrrr|r} x_1 & x_2 & s_1^+ & s_2^+ & s_3^+ & s_1^- & s_2^- & s_3^- & s_4 & \text{rhs} \\ \hline 0 & 0 & 0 & 0 & 0 & P_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & P_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_3 & 0 & 0 \\ \hline 7 & 3 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 40 \\ 10 & 5 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 60 \\ 5 & 4 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 35 \\ 100 & 60 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 600 \end{array}$$

The final tableau in Maple:

$$\left[\begin{array}{ccccccccc|c} x_1 & x_2 & s_1^+ & s_2^+ & s_3^+ & s_1^- & s_2^- & s_3^- & s_4 & \text{rhs} \\ 0 & 0 & 0 & 0 & 0 & P1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & P2 & 0 & 0 & 0 \\ 0 & 0 & 0 & P3 & P3 & 0 & -P3 & 0 & \frac{3}{20}P3 & -5\,P3 \\ \hline 1 & 0 & 0 & -\frac{3}{5} & 0 & 0 & \frac{3}{5} & 0 & -\frac{1}{20} & 6 \\ 0 & 0 & 1 & -\frac{6}{5} & 0 & -1 & \frac{6}{5} & 0 & -\frac{1}{20} & 2 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & -\frac{3}{20} & 5 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & \frac{1}{10} & 0 \end{array}\right]$$

Interpretation: By spending 6 units of advertising on football games (x_1) and nothing on soap operas, we should be able to reach our target audience of high income men and high income women.

Example 4: Color TVs and VCRs

This is a modification of Exercise 3 from the text.

> Highland Appliance must determine how many color TVs and VCRs should be stocked. It costs Highland \$300 to purchase a TV and \$200 to purchase a VCR. A TV requires 3 square yards of storage space, and a VCR requires 1 square yard. A TV will bring \$150 in profit, a VCR \$100. Highland has set the following goals (in order of importance):
>
> - Goal 1: We have \$20,000 to buy TVs and VCR's
> - Goal 2: We need at least \$11,000 in profit
> - Goal 3: We have no more than 200 square yards of storage space.
>
> First, formulate a nonpreemptive goal program that makes the penalty for profit 10 times that of the other two goals, and solve with LINDO. Then make them all equal and re-solve (using LINDO).
>
> Secondly, write a preemptive goal program and solve it with Maple.

(Go to next several pages for the solutions)
FIRST PART OF SOLUTION: Write down the constraints and variables needed:

$$\begin{array}{rrll}
300x_1 & +200x_2 & \leq 20000 & \text{Moolah for Product, Penalize overage} \\
150x_1 & +100x_2 & \geq 11000 & \text{Profit, Penalize underage} \\
3x_1 & +2x_2 & \leq 200 & \text{Storage Space, Penalize overage}
\end{array}$$

Therefore, the variables are:

$$\begin{array}{rrrrrl}
300x_1 & +200x_2 & +s_1^- & -s_1^+ & = 20000 & \text{Bad: } s_1^+ \\
150x_1 & +100x_2 & +s_2^- & -s_2^+ & = 11000 & \text{Bad: } s_2^- \\
3x_1 & +2x_2 & +s_3^- & -s_3^+ & = 200 & \text{Bad: } s_3^+
\end{array}$$

SOLUTION 1 (LINDO, with weighted penalties):

```
min s1p+10s2m+s3p
st
300x1+200x2+s1m-s1p=20000
150x1+100x2+s2m-s2p=11000
3x1+2x2+s3m-s3p=200
```

Output from LINDO:

```
LP OPTIMUM FOUND AT STEP      4
        OBJECTIVE FUNCTION VALUE  2020.000
  VARIABLE        VALUE          REDUCED COST
       S1P      2000.000000         0.000000
       S2M         0.000000         7.980000
       S3P        20.000000         0.000000
        X1        73.333336         0.000000
        X2         0.000000         0.000000
       S1M         0.000000         1.000000
       S2P         0.000000         2.020000
       S3M         0.000000         1.000000
```

SOLUTION 2: LINDO, using equal weights

```
        OBJECTIVE FUNCTION VALUE 1000
  VARIABLE        VALUE          REDUCED COST
       S1P         0.000000         0.510000
       S2M      1000.000000         0.000000
       S3P         0.000000         0.000000
        X1        66.666664         0.000000
        X2         0.000000         0.000000
       S1M         0.000000         0.490000
       S2P         0.000000         1.000000
       S3M         0.000000         1.000000
```

SOLUTION 3: Solve the preemptive program using Maple

Originally, in Maple we'll solve the **maximization** problem:

$$\max -z = -P_1 s_1^+ - P_2 s_2^- - P_3 s_3^+ \quad \Rightarrow \quad -z + P_1 s_1^+ + P_2 s_2^- + P_3 s_3^- = 0$$

which we put down in three goal rows.

x_1	x_2	s_1^-	s_1^+	s_2^-	s_2^+	s_3^-	s_3^+	RHS
0	0	0	P_1	0	0	0	0	0
0	0	0	0	P_2	0	0	0	0
0	0	0	0	0	0	0	P_3	0
300	200	1	-1	0	0	0	0	20,000
150	100	0	0	1	-1	0	0	11,000
3	2	0	0	0	0	1	-1	200

Oh no! I can't use the columns with the P_i as basic variables! What am I to do?

ANSWER: The two phase method. In Maple, we get:

x_1	x_2	s_1^-	s_1^+	s_2^-	s_2^+	s_3^-	s_3^+	a_1	a_2	RHS
0	0	0	0	0	0	0	0	1	1	0
0	0	0	P_1	0	0	0	0	0	0	0
0	0	0	0	P_2	0	0	0	0	0	0
0	0	0	0	0	0	0	P_3	0	0	0
300	200	1	-1	0	0	0	0	1	0	20,000
150	100	0	0	1	-1	0	0	0	0	11,000
3	2	0	0	0	0	1	-1	0	1	200

Then (follow along in Maple to the end):

x_1	x_2	s_1^-	s_1^+	s_2^-	s_2^+	s_3^-	s_3^+	a_1	a_2	RHS
0	0	0	0	0	0	0	0	1	1	0
0	0	0	P_1	0	0	0	0	0	0	0
0	0	$P_2/2$	$-P_2/2$	0	P_2	0	0	$P2/2$	0	$-1000P_2$
0	0	$-P_3/100$	$P_3/100$	0	0	P_3	0	$-P_3/100$	P_3	0
0	0	$1/100$	$-1/100$	0	0	-1	1	$1/100$	-1	0
0	0	$-1/2$	$1/2$	1	-1	0	0	$-1/2$	0	1,000
1	$2/3$	$1/300$	$-1/300$	0	0	0	0	$1/300$	0	$200/3$

Notes:

- We cannot pivot in the s_1^+ column to get a better P_2 result without decreasing our first goal (since P_1 is above $-\frac{1}{2}P_2$). Therefore, goal P_2 is finished. Similarly, we cannot pivot in Column 2 (s_1^-) without decreasing Goal 2. Therefore, Goal 3 is finished.
- The zeros to the right of Goals 1 and 3 mean that they have both been met.
- Goal 2 has not been met.
- Current set of basic variables: x_1, s_2^-, and s_3^+. The full BFS is (in order)

$$x_1 = 66\frac{2}{3} \quad x_2 = 0 \quad s_1^{\pm} = 0 \quad s_2^- = 1000, s_2^+ = 0, s_3^{\pm} = 0$$

Solving TV/VCRs with LINDO

As discussed in the text, we can use LINDO to solve the LP with priorities. We start with the initial tableau, and the highest priority goal. Recall that s_1^+, s_2^- and s_3^+ are the "bad" variables for goals 1, 2, and 3 respectively.

```
min s1p
st 300x1+200x2+s1m-s1p=20000
   150x1+100x2+s2m-s2p=11000
     3x1+  2x2+s3m-s3p=200

LINDO OUTPUT:
        OBJECTIVE FUNCTION VALUE
        1)     0.0000000E+00
  VARIABLE        VALUE          REDUCED COST
       S1P         0.000000          1.000000
        X1         0.000000          0.000000
        X2         0.000000          0.000000
       S1M     20000.000000          0.000000
       S2M     11000.000000          0.000000
       S2P         0.000000          0.000000
       S3M       200.000000          0.000000
       S3P         0.000000          0.000000
```

Now, rerun with the new problem using the second priority:

```
min s2m
st 300x1+200x2+s1m-s1p=20000
   150x1+100x2+s2m-s2p=11000
     3x1+  2x2+s3m-s3p=200
                   s1p=0       ***NOTE THIS NEW LINE!

LINDO OUTPUT:
        OBJECTIVE FUNCTION VALUE
        1)      1000.000
  VARIABLE        VALUE          REDUCED COST
       S2M      1000.000000          0.000000
        X1        66.666664          0.000000
        X2         0.000000          0.000000
       S1M         0.000000          0.500000
       S1P         0.000000          0.000000
       S2P         0.000000          1.000000
       S3M         0.000000          0.000000
       S3P         0.000000          0.000000
```

And finally, rerun with the modified system:

```
min s3p
st 300x1+200x2+s1m-s1p=20000
   150x1+100x2+s2m-s2p=11000
     3x1+  2x2+s3m-s3p=200
                   s1p=0
                   s2m=1000
```

```
LINDO OUTPUT:
        OBJECTIVE FUNCTION VALUE
        1)     0.0000000E+00
  VARIABLE        VALUE          REDUCED COST
       S3P         0.000000          1.000000
        X1        66.666664          0.000000
        X2         0.000000          0.000000
       S1M         0.000000          0.000000
       S1P         0.000000          0.000000
       S2M      1000.000000          0.000000
       S2P         0.000000          0.000000
       S3M         0.000000          0.000000
```

(Same solution as Maple with priorities)