Goal Programming Example

The Dewright company is considering three new products. Goals and penalties are shown in the table:

	Product				
Factor	1	2	3	Goal	Penalty
Profit	12	9	15	≥ 125	5
Employment	5	3	4	$= 40$	$2(+), 4(-)$
Capital	5	7	8	≤ 55	3

Formulate a system for goal programming.

Let x_1, x_2, x_3 be the prod. rates. Then we will

$$\min \ z = 5s_1 + 4s_2 + 2e_2 + 3e_3$$

$$\begin{array}{rrrrr} 12x_1 & +9x_2 & +15x_3 & +s_1 - e_1 & = 125 \\ 5x_1 & +3x_2 & +4x_3 & +s_2 - e_2 & = 40 \\ 5x_1 & +7x_2 & +8x_3 & +s_3 - e_3 & = 55 \end{array}$$

We could easily solve this in LINDO or a spreadsheet.
The Maple worksheet is Example 2 online.

$$x_1 = \frac{100}{7}, \quad x_2 = 0, \qquad x_3 = \frac{145}{7}$$

Goal Programming Example

(Exercise 4, 4.16) A company produces two products, where the labor and profit are:

	Prod 1	Prod 2	Target
Labor (hrs)	4	2	32
Profit ($)	4	2	48
Demand	7	10	

The company incurs a penalty of $1 for each dollar it falls short of the profit. A $2 penalty is incurred for each hour of overtime, and a $1 penalty is incurred for each hour of labor that is not used. A penalty of $5/unit is assessed for any shortfall in meeting demand.
Formulate an LP to minimize the penalty.

SOLUTION: We minimize:

$$z = s_1 + 2e_1 + s_2 + 5s_3 + 5s_4$$

$$\begin{array}{lllll} 4x_1 & +2x_2 & +s_1 & -e_1 & = 32 \text{ Labor} \\ 4x_1 & +2x_2 & +s_2 & -e_2 & = 48 \text{ Profit} \\ x_1 & & +s_3 & -e_3 & = 7 \\ & x_2 & +s_4 & -e_4 & = 10 \end{array}$$

SOLUTION: We minimize:

$$z = s_1 + 2e_1 + s_2 + 5s_3 + 5s_4$$

$$\begin{array}{llllll} 4x_1 & +2x_2 & +s_1 & -e_1 & = 32 & \text{Labor} \\ 4x_1 & +2x_2 & +s_2 & -e_2 & = 48 & \text{Profit} \\ x_1 & & +s_3 & -e_3 & = 7 & \\ & x_2 & +s_4 & -e_4 & = 10 & \end{array}$$

In LINDO, we get

$$x_1 = 7, \qquad x_2 = 10, \qquad e_1 = 16$$

SOLUTION: We minimize:

$$z = s_1 + 2e_1 + s_2 + 5s_3 + 5s_4$$

$$\begin{array}{cccccl} 4x_1 & +2x_2 & +s_1 & -e_1 & = 32 & \text{Labor} \\ 4x_1 & +2x_2 & +s_2 & -e_2 & = 48 & \text{Profit} \\ x_1 & & +s_3 & -e_3 & = 7 & \\ & x_2 & +s_4 & -e_4 & = 10 & \end{array}$$

In LINDO, we get

$$x_1 = 7, \qquad x_2 = 10, \qquad e_1 = 16$$

reasonable considering heavy penalty on not meeting demand.

Goal Programming

Now we'll try the same problem, except that we set the following goals (in order of importance):

- Goal 1: Avoid underutilization of labor.
- Goal 2: Meet demand for product 1.
- Goal 3: Meet demand for product 2.
- Goal 4: Do not use any overtime.

We see profit is no longer a constraint.

Setting up the Tableau:

$$\begin{array}{llllll} 4x_1 & +2x_2 & +s_1 & -e_1 & = 32 \text{ Labor} \\ x_1 & & +s_2 & -e_2 & = 7 \\ & x_2 & +s_3 & -e_3 & = 10 \end{array}$$

Goals:

$$P_1 s_1 \gg P_2 s_2 \gg P_3 s_3 \gg P_4 e_1$$

From our equations, we get the tableau (max):

$$\left[\begin{array}{cc|ccc|ccc|c}
x_1 & x_2 & s_1 & s_2 & s_3 & e_1 & e_2 & e_3 & \text{rhs} \\
\hline
0 & 0 & P_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & P_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & P_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & P_4 & 0 & 0 & 0 \\
\hline
4 & 2 & 1 & 0 & 0 & -1 & 0 & 0 & 32 \\
1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 7 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 10
\end{array}\right]$$

Choose a set of variables for the BFS, and row reduce to get columns of the identity matrix.

From our equations, we get the tableau (max):

$$\left[\begin{array}{cc|ccc|ccc|c}
x_1 & x_2 & s_1 & s_2 & s_3 & e_1 & e_2 & e_3 & \text{rhs} \\
\hline
0 & 0 & P_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & P_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & P_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & P_4 & 0 & 0 & 0 \\
\hline
4 & 2 & 1 & 0 & 0 & -1 & 0 & 0 & 32 \\
1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 7 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 10
\end{array}\right]$$

Choose a set of variables for the BFS, and row reduce to get columns of the identity matrix.

Current basic variables: $s_1 = 32, s_2 = 7, s_3 = 10$.

Clearing those columns, here is the result:

x_1	x_2	s_1	s_2	s_3	e_1	e_2	e_3	rhs
$-4P_1$	$-2P_1$	0	0	0	P_1	0	0	$-32P_1$
$-P_2$	0	0	0	0	0	P_2	0	$-7P_2$
0	$-P_3$	0	0	0	0	0	P_3	$-10P_3$
0	0	0	0	0	P_4	0	0	0
4	2	1	0	0	-1	0	0	32
1	0	0	1	0	0	-1	0	7
0	1	0	0	1	0	0	-1	10

Using the first row as Row 0, pivot in column 1, row 6 (use the ratio test).
This will bring in x_1 as BV, and move s_2 to NBV.

x_1	x_2	s_1	s_2	s_3	e_1	e_2	e_3	rhs
0	$-2P_1$	0	$4P_1$	0	P_1	$-4P_1$	0	$-4P_1$
0	0	0	P_2	0	0	0	0	0
0	$-P_3$	0	0	0	0	0	P_3	$-10P_3$
0	0	0	0	0	P_4	0	0	0
0	2	1	-4	0	-1	4	0	4
1	0	0	1	0	0	-1	0	7
0	1	0	0	1	0	0	-1	10

Now bring in e_2 as basic, and move s_1 to NBV.
That is, pivot in Column 7, Row 5.

x_1	x_2	s_1	s_2	s_3	e_1	e_2	e_3	rhs
0	0	P_1	0	0	0	0	0	0
0	0	0	P_2	0	0	0	0	0
0	$-P_3$	0	0	0	0	0	P_3	$-10P_3$
0	0	0	0	0	P_4	0	0	0
0	$1/2$	$1/4$	-1	0	$-1/4$	1	0	1
1	$1/2$	$1/4$	0	0	$-1/4$	0	0	8
0	1	0	0	1	0	0	-1	10

Goals 1 and 2 now complete. Proceed to Priority 3.
Bring in x_2 as basic, take e_2 back out again.

$$\left[\begin{array}{cc|ccc|ccc|c}
x_1 & x_2 & s_1 & s_2 & s_3 & e_1 & e_2 & e_3 & \text{rhs} \\
\hline
0 & 0 & P_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & P_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & P_3/2 & -2P_3 & 0 & -P_3/2 & 2P_3 & P_3 & -8P_3 \\
0 & 0 & 0 & 0 & 0 & P_4 & 0 & 0 & 0 \\
\hline
0 & 1 & 1/2 & -2 & 0 & -1/2 & 2 & 0 & 2 \\
1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 7 \\
0 & 0 & -1/2 & 2 & 1 & 1/2 & -2 & -1 & 8
\end{array}\right]$$

We can't bring in s_2 without messing up P_2. Are we done?

$$
\left[\begin{array}{cc|ccc|ccc|c}
x_1 & x_2 & s_1 & s_2 & s_3 & e_1 & e_2 & e_3 & \text{rhs} \\
\hline
0 & 0 & P_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & P_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & P_3/2 & -2P_3 & 0 & -P_3/2 & 2P_3 & P_3 & -8P_3 \\
0 & 0 & 0 & 0 & 0 & P_4 & 0 & 0 & 0 \\
\hline
0 & 1 & 1/2 & -2 & 0 & -1/2 & 2 & 0 & 2 \\
1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 7 \\
0 & 0 & -1/2 & 2 & 1 & 1/2 & -2 & -1 & 8
\end{array}\right]
$$

We can't bring in s_2 without messing up P_2. Are we done? Bring in e_1, take out s_3.

$$
\left[\begin{array}{cc|ccc|ccc|c}
x_1 & x_2 & s_1 & s_2 & s_3 & e_1 & e_2 & e_3 & \text{rhs} \\
\hline
0 & 0 & P_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & P_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & P_3 & 0 & 0 & 0 & 0 \\
0 & 0 & P_4 & -4P_4 & -2P_4 & 0 & 4P_4 & 2P_4 & -16P_4 \\
\hline
0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 10 \\
1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 7 \\
0 & 0 & -1 & 4 & 2 & 1 & -4 & -2 & 16
\end{array}\right]
$$

Optimal?

$$
\left[\begin{array}{cc|ccc|ccc|c}
x_1 & x_2 & s_1 & s_2 & s_3 & e_1 & e_2 & e_3 & \text{rhs} \\
\hline
0 & 0 & P_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & P_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & P_3 & 0 & 0 & 0 & 0 \\
0 & 0 & P_4 & -4P_4 & -2P_4 & 0 & 4P_4 & 2P_4 & -16P_4 \\
\hline
0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 10 \\
1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 7 \\
0 & 0 & -1 & 4 & 2 & 1 & -4 & -2 & 16
\end{array}\right]
$$

Optimal?

$$
x_1 = 7, \qquad x_2 = 10, \qquad e_1 = 16
$$

Solving in LINDO

Let's look at solving the goal programming LP in LINDO.
We first ask LINDO to min $z = s_1$ such that

$$\begin{aligned} 4x_1 + 2x_2 + s_1 - e_1 &= 32 \\ x_1 + s_2 - e_2 &= 7 \\ x_2 + s_3 - e_3 &= 10 \end{aligned}$$

Type the following into LINDO, then "Solve":

```
min s1
st
4x1+2x2+s1-e1=32
x1+s2-e2=7
x2+s3-e3=10
end
```

LINDO returns: $x_1 = 7$ and $x_2 = 2$.

Now we ask LINDO to solve the following system (just make the changes “live”)

```
min s2
st
4x1+2x2+s1-e1=32
x1+s2-e2=7
x2+s3-e3=10
s1=0
end
```

LINDO returns: $x_1 = 8$ and $x_2 = 0$.

Continuing, we have:

```
min s3
st
4x1+2x2+s1-e1=32
x1+s2-e2=7
x2+s3-e3=10
s1=0
s2=0
end
```

LINDO returns: $x_1 = 7$ and $x_2 = 10$.

Finally, we ask LINDO:

```
min e1
st
4x1+2x2+s1-e1=32
x1+s2-e2=7
x2+s3-e3=10
s1=0
s2=0
s3=0
end
```

LINDO basically returns the same answer:

$$x_1 = 7, \quad x_2 = 10, \quad e_1 = 16$$

(So we're not able to get e_1 to zero).