Section 4.5: Second Example in Detail

Recall the Simplex Algorithm:

1. Put the LP in standard form, then build the simplex tableau (as a max problem).
2. Find an initial BFS.
3. Look at Row 0. The column corresponding to the largest negative number will represent the variable that we want to bring into the set of basic variables. This is the pivot column.
4. Look at the RHS, and compute the ratio of the RHS over the corresponding coefficient in the pivot column. This number represents how much we can increase the new variable by, and still keep that constraint consistent.
5. Choose the row with the smallest ratio. This is the pivot element.
6. Perform the row operations to zero out all elements above and below the pivot element (and make the pivot element 1).
7. Repeat from Step 3 until either:
   - There are no more negative coefficients. The current solution is the optimal solution.
   - We'll look at other possibilities in 4.6 (and in the example below).

Example from Class

$$\begin{array}{rrl} \min & 2x_1 + x_2 - 4x_3 & \\ \text{st} & 3x_1 - x_2 + 2x_3 & \leq 25 \\ & -x_1 - x_2 + 2x_3 & \leq 20 \\ & -x_1 - x_2 + x_3 & \leq 5 \end{array} \quad \Rightarrow \quad \begin{array}{rrl} \max & -2x_1 - x_2 + 4x_3 & \\ \text{st} & 3x_1 - x_2 + 2x_3 & \leq 25 \\ & -x_1 - x_2 + 2x_3 & \leq 20 \\ & -x_1 - x_2 + x_3 & \leq 5 \end{array}$$

with all variables non-negative. We changed this to a maximization problem by multiplying by -1 (the array above right). Here's the initial tableau:

$$\begin{array}{c|rrrrrr|r} z & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & rhs \\ 1 & 2 & 1 & -4 & 0 & 0 & 0 & 0 \\ \hline 0 & 3 & -1 & 2 & 1 & 0 & 0 & 25 \\ 0 & -1 & -1 & 2 & 0 & 1 & 0 & 20 \\ 0 & -1 & -1 & 1 & 0 & 0 & 1 & 5 \end{array}$$

This corresponds to an initial BFS using s_1, s_2 and s_3 as the basic variables. The first BFS is thus $\mathbf{x} = [0, 0, 0, 25, 20, 5]^T$. Next, look for the column with the most negative number. In this case, that is the x_3 column, and that is our pivot column.

To determine the pivot row, we look at the ratios of the RHS column to the coefficients in the x_3 column. In this case, they are, in order: $25/2, 20/2$ and $5/1$. Choose the smallest, which is the third row. That gives our pivot position as $(3, 3)$ (remember the first row is row 0).

$$\begin{array}{rrrrrrr|l}
2 & 1 & -4 & 0 & 0 & 0 & 0 & \\
\hline
3 & -1 & 2 & 1 & 0 & 0 & 25 & 25/2 \\
-1 & -1 & 2 & 0 & 1 & 0 & 20 & 20/2 = 10 \\
-1 & -1 & 1 & 0 & 0 & 1 & 5 & 5/1 = 5
\end{array}
\quad \Rightarrow \quad
\begin{array}{rrrrrr|r}
-2 & -3 & 0 & 0 & 0 & 4 & 20 \\
\hline
5 & 1 & 0 & 1 & 0 & -2 & 15 \\
1 & 1 & 0 & 0 & 1 & -2 & 10 \\
-1 & -1 & 1 & 0 & 0 & 1 & 5
\end{array}$$

This tableau gives the new BFS: $\mathbf{x} = [0, 0, 5, 15, 10, 0]^T$ and the current value of z is 20.

Now we repeat the process: Look for the most negative coefficient in Row 0, and that will be our next pivot column. In this case, x_2.

When computing which row to use for the pivot, we compute the usual ratios. We run into a problem now- Here's the rule: **In the "ratio test", we ignore division by a negative number and division by zero.** Why? Consider the coefficients in the last row, and the associated equation. Writing it out and keeping x_3 basic (so $x_3 = 5$), then:

$$[-1, -1, 1, 0, 0, 1, 5] \quad \Rightarrow \quad S_3 = 5 + x_1 + x_2 - x_3 = x_1 + x_2$$

Does this equation provide any constraint on x_2? No- We can increase x_2 without bound and *this* constraint would be satisfied. That's why we ignore this equation in the ratio computations.

Getting back to the algorithm: the other ratios are 15 or 10, so use Row 2, and the pivot position is $(2, 2)$. The result of row reduction is:

$$\begin{array}{rrrrrr|r}
1 & 0 & 0 & 0 & 3 & -2 & 50 \\
\hline
4 & 0 & 0 & 1 & -1 & 0 & 5 \\
1 & 1 & 0 & 0 & 1 & -2 & 10 \\
0 & 0 & 1 & 0 & 1 & -1 & 15
\end{array}$$

This corresponds to the BFS: $\mathbf{x} = [0, 10, 15, 5, 0, 0]^T$. Can we continue? Normally, we would choose to bring s_3 back in (it is the only column with a leading negative value). Does this make sense in terms of our optimization? The ratios are:

$$\frac{5}{0} \qquad \frac{10}{-2} \qquad \frac{15}{-1}$$

so I cannot bring this variable in... Convert our tableau back into a system of equations to see what is happening. We'll keep the NBV as free variables:

$$\begin{array}{rllllll}
\max & z & = 50 & -x_1 & -3s_2 & +2s_3 \\
\text{st} & s_1 & = 5 & -4x_1 & +s_2 & \\
 & x_2 & = 10 & -x_1 & -s_2 & +2s_3 \\
 & x_3 & = 15 & & -s_2 & +s_3
\end{array}$$

Conclusion: There is no restriction on how large s_3 can be- The objective function is unbounded. To tie this up, the full set of variables can be solved for:

$$\begin{array}{lllll}
x_1 & = & x_1 & & \\
x_2 & = 10 & -x_1 & -s_2 & +2s_3 \\
x_3 & = 15 & & -s_2 & +s_3 \\
s_1 & = 5 & -4x_1 & +s_2 & \\
s_2 & = & & s_2 & \\
s_3 & = & & & s_3
\end{array}$$

Notice that the following vectors form a basis for the null space of A.

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ -4 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The last column forms a direction of unboundedness. If we check, $\mathbf{c}$ is not orthogonal to $\mathbf{d}$. Is that a problem?