We're buying advertising time for "HIW" and "HIM".
Let x_1, x_2 be the number of ads purchased during a comedy show, and a football game respectively. Comedy ads are \$50,000 and football ads are \$100,000. For the target demos, we want to solve the following LP:

$$\begin{array}{rll} \min z = & 50x_1 + 100x_2 & \\ \text{st} & 7x_1 + 2x_2 \geq 28 & \text{(HIW)} \\ & 2x_1 + 12x_2 \geq 24 & \text{(HIM)} \\ & x_1, x_2 \geq 0 & \end{array}$$

where the unit of money is \$1,000 and the unit of people is in millions.

To solve this LP, we would need big-M or two phase. If we do that, here is the optimal tableau:

x_1	x_2	e_1	e_2	rhs
50	100	0	0	0
7	2	-1	0	28
2	12	0	-1	24

$\Rightarrow$

x_1	x_2	e_1	e_2	rhs
0	0	5	$\frac{15}{2}$	-320
1	0	$-\frac{3}{20}$	$\frac{1}{40}$	$\frac{18}{5}$
0	1	$\frac{1}{40}$	$-\frac{7}{80}$	$\frac{7}{5}$

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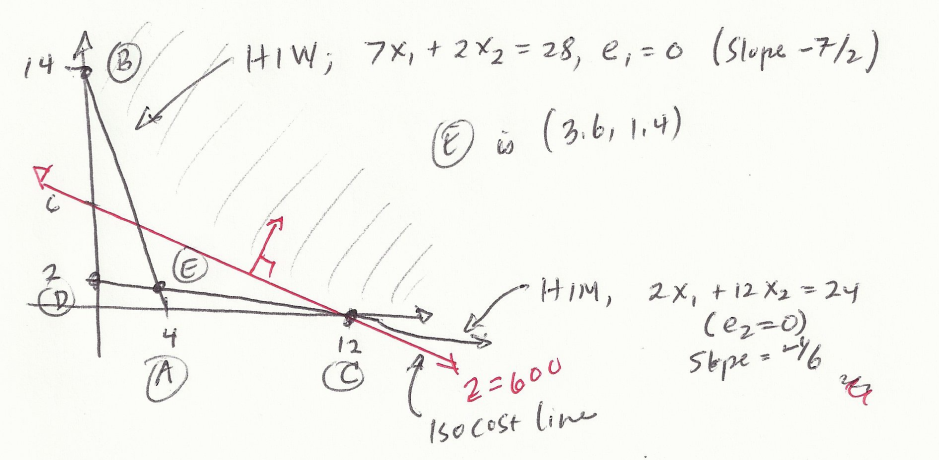
On the next page, the solution is done graphically.


14
B
H1W; 7x₁ + 2x₂ = 28, e₁ = 0 (Slope −7/2)
E is (3.6, 1.4)
6
2
E
D
4
12
A
C
H1M, 2x₁ + 12x₂ = 24
(e₂=0)
Slope = −1/6
z=600
Isocost line

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$\Rightarrow$

x_1	x_2	e_1	e_2	rhs
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Isocost line: $C_1 x_1 + 100 x_2 = k$, so the slope is

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2	12	0	-1	24

Isocost line: $C_1x_1 + 100x_2 = k$, so the slope is $-C_1/100$.
Therefore, the current basis stays optimal if:

$$-\frac{7}{2} < -\frac{C_1}{100} < -\frac{1}{6} \quad \Rightarrow \quad \frac{50}{3} < C_1 < 350$$

By how much could the price of the football ads change, and still give the same optimal solution?

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x_1	x_2	e_1	e_2	rhs
50	100	0	0	0
7	2	-1	0	28
2	12	0	-1	24

$\Rightarrow$

x_1	x_2	e_1	e_2	rhs
50	C_2	0	0	0
7	2	-1	0	28
2	12	0	-1	24

Isocost line: $50x_1 + C_2x_2 = k$, so the slope is

By how much could the price of the football ads change, and still give the same optimal solution?

x_1	x_2	e_1	e_2	rhs
50	100	0	0	0
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2	12	0	-1	24

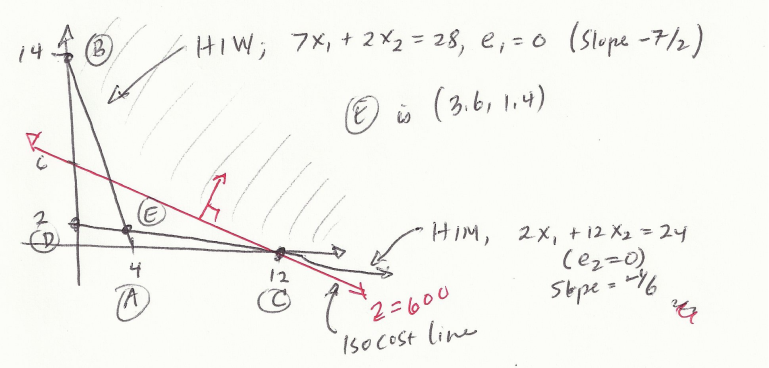
$\Rightarrow$

x_1	x_2	e_1	e_2	rhs
50	C_2	0	0	0
7	2	-1	0	28
2	12	0	-1	24

Isocost line: $50x_1 + C_2x_2 = k$, so the slope is $-50/C_2$.
Therefore, the current basis stays optimal if:

$$-\frac{7}{2} < -\frac{50}{C_2} < -\frac{1}{6} \quad \Rightarrow \qquad \Rightarrow \quad \frac{100}{7} \leq C_2 \leq 300$$

What happens if we can change the RHS of constraint 1? Consider the graph first, and change 28 to b_1:



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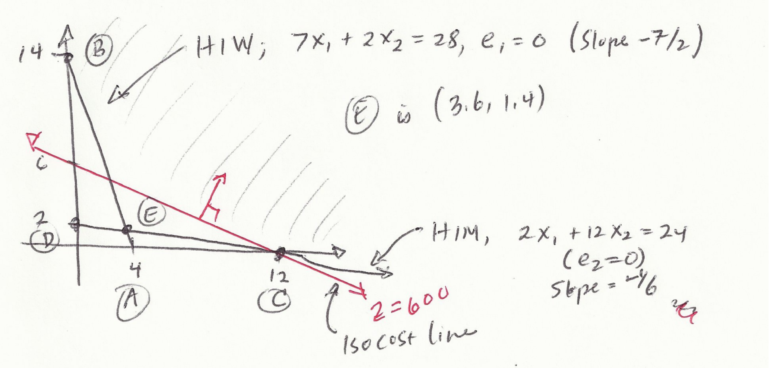

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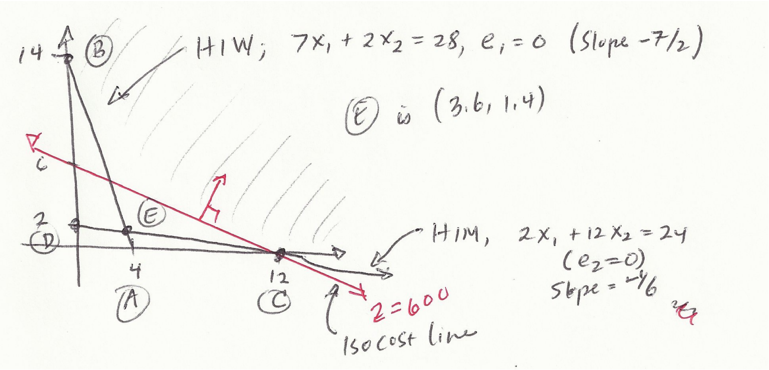

Note that the numerical value of the solution changes-Does the *basis* change? For what values of b_1?

x_1	x_2	e_1	e_2	rhs
50	100	0	0	0
7	2	-1	0	b_1
2	12	0	-1	24

Current basis is optimal if E remains at the optimum. As b_1 decreases, point E slides into point D. The b_1 value there is

$$7(0) + 2(2) = 4 \quad \Rightarrow \quad b_1 > 4$$

Similarly, if b_1 moves up, then E slides into C. This occurs when

$$7(12) + 2(0) = 84 \quad \Rightarrow \quad b_1 < 84$$

Or, if $b_1 = 28 + \Delta_1$, then we could say that

$$4 < 28 + \Delta_1 < 84 \quad \Rightarrow \quad -24 < \Delta_1 < 56$$

We solve for the point of intersection of the system below using Cramer's Rule. Using the current basis, x_1 and x_2 are basic and e_1, e_2 are set to zero.

$$\begin{aligned} 7x_1 + 2x_2 &= 28 + \Delta_1 \\ 2x_1 + 12x_2 &= 24 \end{aligned}$$

$$x_1 = \frac{\begin{vmatrix} 28+\Delta_1 & 2 \\ 24 & 12 \end{vmatrix}}{\begin{vmatrix} 7 & 2 \\ 2 & 12 \end{vmatrix}} = \frac{336 + 12\Delta_1 - 48}{7 \cdot 12 - 4} = 3.6 + 0.15\Delta_1$$

$$x_2 = \frac{\begin{vmatrix} 7 & 28+\Delta_1 \\ 2 & 24 \end{vmatrix}}{\begin{vmatrix} 7 & 2 \\ 2 & 12 \end{vmatrix}} = \frac{168 - 56 - 2\Delta_1}{7 \cdot 12 - 4} = 1.4 - 0.025\Delta_1$$

Similarly, for the HIM constraint, $2x_1 + 12x_2 = b_2$:

- As b_2 increases, point E slides toward $B(0, 14)$.

$$2(0) + 12(14) = b_2 \quad \Rightarrow \quad b_2 = 168$$

- As b_2 decreases, point E slides toward $A(4, 0)$.

$$2(4) + 12(0) = b_2 \quad \Rightarrow \quad b_2 = 8$$

Therefore,

$$8 \leq b_2 \leq 168$$

So, in terms of Δ_2, we could compute the range directly:

$$8 \leq 24 + \Delta_2 \leq 168 \quad \Rightarrow \quad -16 \leq \Delta_2 \leq 144$$

And for the point of intersection between the constraints (note the basis)

$$\begin{aligned} 7x_1 + 2x_2 &= 28 \\ 2x_1 + 12x_2 &= 24 + \Delta_2 \end{aligned}$$

$$x_1 = \frac{\begin{vmatrix} 28 & 2 \\ 24 + \Delta_2 & 12 \end{vmatrix}}{\begin{vmatrix} 7 & 2 \\ 2 & 12 \end{vmatrix}} = \frac{336 - 48 - 2\Delta_2}{80} = 3.6 - 0.025\Delta_2$$

And

$$x_2 = \frac{\begin{vmatrix} 7 & 28 \\ 2 & 24 + \Delta_2 \end{vmatrix}}{\begin{vmatrix} 7 & 2 \\ 2 & 12 \end{vmatrix}} = \frac{168 - 56 + 7\Delta_2}{80} = 1.4 + 0.0875\Delta_2$$

Definition: Shadow Price

The shadow price (of a constraint) is the amount by which the value of the objective function *is improved* (increased in a maximum, decreased in a minimum) when the RHS of the constraint is *increased by 1 unit*. This is done *under the assumption that the current basis is unchanged*.

Example: Find the Shadow Prices

For *HIW*, if the RHS is $28 + \Delta$, then we solve the system below, which we already did:

$$\begin{aligned} 7x_1 + 2x_2 &= 28 + \Delta \\ 2x_1 + 12x_2 &= 24 \end{aligned} \quad \Rightarrow \quad x_1 = 3.6 + 0.15\Delta, x_2 = 1.4 - 0.025\Delta$$

Substitute these into the objective function:

$$(\min) z = 50(3.6 + 0.15\Delta) + 100(1.4 - 0.025\Delta) = 320 + 5\Delta$$

What is the shadow price for HIW?

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What is the shadow price for HIW?

The shadow price is -5 (this is a minimization problem)

For HIM, if the RHS is $24 + \Delta$, then we solve the system below:

$$\begin{aligned} 7x_1 + 2x_2 &= 28 \\ 2x_1 + 12x_2 &= 24 + \Delta \end{aligned} \quad \Rightarrow \quad x_1 = 3.6 - 0.025\Delta, x_2 = 1.4 + 0.0875\Delta$$

Substitute these into the objective function:

$$(\min) z = 50(3.6 - 0.025\Delta) + 100(1.4 + 0.0875\Delta) = 320 + 7.5\Delta$$

For HIM, if the RHS is $24 + \Delta$, then we solve the system below:

$$\begin{aligned} 7x_1 + 2x_2 &= 28 \\ 2x_1 + 12x_2 &= 24 + \Delta \end{aligned} \quad \Rightarrow \quad x_1 = 3.6 - 0.025\Delta, x_2 = 1.4 + 0.0875\Delta$$

Substitute these into the objective function:

$$(\text{min})z = 50(3.6 - 0.025\Delta) + 100(1.4 + 0.0875\Delta) = 320 + 7.5\Delta$$

The shadow price is -7.5.

Extra Example:

If 26 HIW exposures are required (and other parameters remain the same), determine the new solution and the new value of z.

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SOLUTION: Just set $\Delta = -2$ (which is in the allowable range). Then the optimal solution changes slightly:

$$x_1 = 3.6 + 0.15(-2) = 3.30 \qquad x_2 = 1.4 - 0.025(-2) = 1.45$$

and $z = 320 + 5(-2) = 310$ (double-check: $50(3.3) + 100(1.45) = 310$)

Solving this problem in LINDO: Choose "yes" when LINDO asks you if you want to do sensitivity analysis:

```
min 50x1+100x2
st
(HIW) 7x1+2x2>=28
(HIM) 2x1+12x2>=24
```

```
LP OPTIMUM FOUND AT STEP      2

        OBJECTIVE FUNCTION VALUE

        1)      320.0000

  VARIABLE        VALUE          REDUCED COST
        X1         3.600000          0.000000
        X2         1.400000          0.000000


       ROW   SLACK OR SURPLUS     DUAL PRICES
     (HIW)         0.000000         -5.000000
     (HIM)         0.000000         -7.500000
```

NO. ITERATIONS= 2

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
X1	50.000000	300.000000	33.333332
X2	100.000000	200.000000	85.714287

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
(HIW	28.000000	56.000000	23.999998
(HIM	24.000000	144.000000	16.000000