Extra Worked Example: Full Sensitivity Analysis

A factory can produce 4 products. Each product must be processed in each of two workshops. The processing times and profit margins for each of the four products is shown.

	1	2	3	4
Workshop 1	3	4	8	6
Workshop 2	6	2	5	8
Profit	4	6	10	9

If we have 400 hours of labor available in each workshop, the following LP can be used:

$$\begin{array}{llrrrrl} \max & z = & 4x_1 & +6x_2 & +10x_3 & +9x_4 & \\ \text{st} & & 3x_1 & +4x_2 & +8x_3 & +6x_4 & \leq 400 \text{ Labor 1} \\ & & 6x_1 & +2x_2 & +5x_3 & +8x_4 & \leq 400 \text{ Labor 2} \end{array}$$

The initial and final tableaux:

x_1	x_2	x_3	x_4	s_1	s_2	
-4	-6	-10	-9	0	0	0
3	4	8	6	1	0	400
6	2	5	8	0	1	400

x_1	x_2	x_3	x_4	s_1	s_2	
1/2	0	2	0	3/2	0	600
3/4	1	2	3/2	1/4	0	100
9/2	0	1	5	$-1/2$	1	200

1. Sensitivity Analysis on the NBVs.

- x_1: Change 4 to $4+\Delta$.
  Discussion: This changes only $\mathbf{c}_1$. In this case,

  $$\hat{c}_1 = -(\mathbf{c}_1^T + \Delta) + (\mathbf{c}_B^T B^{-1} A)_1 = (-\mathbf{c}^T + \mathbf{c}_B^T B^{-1} A)_1 - \Delta = \frac{1}{2} - \Delta$$

  Therefore, Δ may increase only to 1/2, and may decrease to any amount.
- x_3: Change 10 to $10+\Delta$.
  Discussion: This changes only $\mathbf{c}_3$. In this case,

  $$\hat{c}_3 = -(\mathbf{c}_3^T + \Delta) + (\mathbf{c}_B^T B^{-1} A)_3 = (-\mathbf{c}^T + \mathbf{c}_B^T B^{-1} A)_3 - \Delta = 2 - \Delta$$

  Therefore, Δ may increase only to 2, and may decrease to any amount.
- x_4: Change 9 to $9+\Delta$.
  Discussion: This changes only $\mathbf{c}_4$. In this case,

  $$\hat{c}_4 = -(\mathbf{c}_4^T + \Delta) + (\mathbf{c}_B^T B^{-1} A)_4 = (-\mathbf{c}^T + \mathbf{c}_B^T B^{-1} A)_4 - \Delta = -\Delta$$

  Therefore, Δ must be less than 0, and may decrease to any amount.
- We could also ask change the value of s_1 (in z). By the same reasoning of the previous variables, we would get $\frac{3}{2} - \Delta$.

2. Sensitivity of BVs.

   - Change x_2 from 6 to $6+\Delta$.

     Discussion: This changes $\mathbf{c}$ and $\mathbf{c}_B$:

     $$-(\mathbf{c}^T + \Delta\vec{e}_2) + (\mathbf{c}_B + \Delta\vec{e}_1)^T B^{-1}A =$$

     $$(-\mathbf{c}^T + \mathbf{c}_B^T B^{-1}A) - \Delta\vec{e}_2 + \Delta[1,0]B^{-1}A$$

     The first expression is the original final Row 0. The last expression is Δ times the first row of the final tableau. Writing this as a sum of three row vectors:

     $$\begin{array}{rrrrrrr} & [1/2 & 0 & 2 & 0 & 3/2 & 0] \\ - & [\ \ 0 & \Delta & 0 & 0 & 0 & 0] \\ + & \Delta[3/4 & 1 & 2 & 3/2 & 1/4 & 0] \\ \hline & [\frac{1}{2}+\frac{3}{4}\Delta & 0 & 2+2\Delta & \frac{3}{2}\Delta & \frac{3}{2}+\frac{1}{4}\Delta & 0] \end{array}$$

     We want all four non-zero expressions to be non-negative. Take the intersection of the four intervals, and we should see in this case that $\Delta \geq 0$ will satisfy all four.

   - Change s_2 from 0 to Δ.

     Discussion: This changes $\mathbf{c}$ and $\mathbf{c}_B$:

     $$-(\mathbf{c}^T + \Delta\vec{e}_6) + (\mathbf{c}_B + \Delta\vec{e}_2)^T B^{-1}A =$$

     $$(-\mathbf{c}^T + \mathbf{c}_B^T B^{-1}A) - \Delta\vec{e}_6 + \Delta[0,1]B^{-1}A$$

     The first expression is the original final Row 0. The last expression is Δ times the second row of the final tableau. Writing this as a sum of three row vectors:

     $$\begin{array}{rrrrrrr} & [1/2 & 0 & 2 & 0 & 3/2 & 0] \\ - & [\ \ 0 & 0 & 0 & 0 & 0 & \Delta] \\ + & \Delta[9/2 & 0 & 1 & 5 & -1/2 & 1] \\ \hline & [\frac{1}{2}+\frac{9}{2}\Delta & 0 & 2+\Delta & 5\Delta & \frac{3}{2}-\frac{1}{2}\Delta & 0] \end{array}$$

     We want all four non-zero expressions to be non-negative. Take the intersection of the four intervals, and we should see:

     $$0 \leq \Delta \leq 3$$

3. Changes in the RHS and the Shadow Prices.

   - Change in the first constraint: $\mathbf{b}$ changes to $\mathbf{b}+\Delta\vec{e}_1$, so the RHS changes to:

     $$B^{-1}(\mathbf{b}+\Delta\vec{e}_1) = B^{-1}\mathbf{b} + \Delta B_1^{-1}$$

where B_1^{-1} is the first column of B^{-1}. Using our numbers, we get that the RHS changes to:

$$\begin{bmatrix} 100 \\ 200 \end{bmatrix} + \Delta \begin{bmatrix} 1/4 \\ -1/2 \end{bmatrix} \quad \Rightarrow \quad z = 6(100 + \Delta/4) = 600 + \frac{3}{2}\Delta$$

The shadow price for the first constraint is 3/2. We can also compute bounds on Δ so that the new RHS stays non-negative.

- Change in the second constraint. Using a similar computation, we get:

$$B^{-1}\mathbf{b} + \Delta B_2^{-1} = \begin{bmatrix} 100 \\ 200 \end{bmatrix} + \Delta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

  so that $z = 600$. The shadow price is 0. We can also compute bounds on Δ so that the new RHS stays non-negative.

  NOTE: It makes sense that the shadow price is zero- In the optimal tableau, if $s_2 = 200$, then we have an extra 200 hours of labor available. Increasing that by 1 does nothing to z.