Example

$$\begin{array}{rll} \max & 6x_1 & +5x_2 \\ \text{s.t.} & x_1 & +x_2 \leq 5 \\ & 3x_1 & +2x_2 \leq 12 \end{array}$$

with $x_1, x_2 \geq 0$. Find an upper bound to the maximum.

- Using constraint 2 (multiply by 3):

$$6x_1 + 5x_2 \leq 9x_1 + 6x_2 \leq 36$$

- Using constraint 1 (multiply by 6):

$$6x_1 + 5x_2 \leq 6x_1 + 6x_2 \leq 30$$

Example

$$\begin{array}{rll} \max & 6x_1 & +5x_2 \\ \text{s.t.} & x_1 & +x_2 \leq 5 \\ & 3x_1 & +2x_2 \leq 12 \end{array}$$

with $x_1, x_2 \geq 0$. Find an upper bound to the maximum.

We could take a linear combination of the constraints:

$$\begin{array}{rrl} & x_1 + x_2 & \leq 5 \\ + & 2(3x_1 + 2x_2) & \leq 12 \cdot 2 \\ \hline & 7x_1 + 5x_2 & \leq 29 \end{array}$$

More generally:

$$
\begin{array}{rrl}
y_1(x_1 & +x_2) & \leq 5y_1 \\
+y_2(3x_1 & +2x_2) & \leq 12y_2 \\
\hline
(y_1 + 3y_2)x_1 & +(y_1 + 2y_2)x_2 & \leq 5y_1 + 12y_2
\end{array}
$$

This gives us **The Dual**:

$$
\begin{array}{rl}
\min & 5y_1 + 12y_2 \\
\text{s.t.} & y_1 + 3y_2 \geq 6 \\
 & y_1 + 2y_2 \geq 5
\end{array}
$$

with $y_1, y_2 \geq 0$.

We showed that the primal, dual are:

$$\begin{array}{rlll} \max & 6x_1 & +5x_2 \\ \text{s.t.} & x_1 & +x_2 \le 5 \\ & 3x_1 & +2x_2 \le 12 \end{array} \iff \begin{array}{rl} \min & 5y_1 + 12y_2 \\ \text{s.t.} & y_1 + 3y_2 \ge 6 \\ & y_1 + 2y_2 \ge 5 \end{array}$$

So what would it be for the "normal" max problem:

$$\begin{array}{rl} \max z = & \mathbf{c}^T\mathbf{x} \\ \text{st} & A\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge 0 \end{array} \iff \begin{array}{rl} \min w = & \mathbf{b}^T\mathbf{y} \\ \text{st} & A^T\mathbf{y} \ge \mathbf{c} \\ & \mathbf{y} \ge 0 \end{array}$$

Note: **b** may have negative values in this construction.

$$\begin{aligned} \max z = \;& \mathbf{c}^T\mathbf{x} \\ \text{st} \;& A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \min w = \;& \mathbf{b}^T\mathbf{y} \\ \text{st} \;& A^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq 0 \end{aligned}$$

Notes:

- Every primal has a dual.
- A maximum becomes a minimum (dual of dual is primal).
- Obj Func Coeffs in primal converted to RHS of constraints in dual.
- RHS of Constraints in primal converted to Obj Func Coeffs in dual.

An example with non-"normal" issues. Find the dual for:

$$
\begin{aligned}
\min \quad & 8x_1 + 5x_2 + 4x_3 \\
\text{s.t.} \quad & 4x_1 + 2x_2 + 8x_3 = 12 \\
& 7x_1 + 5x_2 + 6x_3 \geq 9 \\
& 8x_1 + 5x_2 + 4x_3 \leq 10 \\
& 3x_1 + 7x_2 + 9x_3 \geq 7 \\
& x_1 \geq 0, x_2 \text{ URS}, x_3 \leq 0
\end{aligned}
$$

We'll put this system into "normal form":

- Change the min to a max.
- Multiply Constraints 2 and 4 by -1.
- Change $=$ into inequalities.
- Find substitutions for x_2, x_3.

First three issues:

$$
\begin{array}{lrrrl}
\max & -8x_1 & -5x_2 & -4x_3 & \\
\text{s.t.} & 4x_1 & +2x_2 & +8x_3 & \leq 12 \\
 & -4x_1 & -2x_2 & -8x_3 & \leq -12 \\
 & -7x_1 & -5x_2 & -6x_3 & \leq -9 \\
 & 8x_1 & +5x_2 & +4x_3 & \leq 10 \\
 & -3x_1 & -7x_2 & -9x_3 & \leq -7
\end{array}
$$

$$x_1 \geq 0, x_2 \text{ URS}, x_3 \leq 0$$

Last issue: Let $x_2 = x_4 - x_5$ and $x_3 = -x_6$ (with all these new vars non-neg)

We now have a "normal" primal:

$$
\begin{array}{llllll}
\max & -8x_1 & -5x_4 & +5x_5 & +4x_6 & \\
\text{s.t.} & 4x_1 & +2x_4 & -2x_5 & -8x_6 & \leq 12 \\
 & -4x_1 & -2x_4 & +2x_5 & +8x_6 & \leq -12 \\
 & -7x_1 & -5x_4 & +5x_5 & +6x_6 & \leq -9 \\
 & 8x_1 & +5x_4 & -5x_5 & -4x_6 & \leq 10 \\
 & -3x_1 & -7x_4 & +7x_5 & +9x_6 & \leq -7
\end{array}
$$

$$x_1, x_4, x_5, x_6 \geq 0$$

From which we get the dual.
Let $p_1, \ldots, p_5$ be the new vars. Using the "normal" primal to dual conversion, we get the following for the dual:

$$\begin{array}{rrrrrrl} \min w = & 12p_1 & -12p_2 & -9p_3 & +10p_4 & -7p_5 & \\ \text{st} & 4p_1 & -4p_2 & -7p_3 & +8p_4 & -3p_5 & \geq -8 \\ & 2p_1 & -2p_2 & -5p_3 & +5p_4 & -7p_5 & \geq -5 \\ & -2p_1 & +2p_2 & +5p_3 & -5p_4 & +7p_5 & \geq 5 \\ & -8p_1 & +8p_2 & +6p_3 & -4p_4 & +9p_5 & \geq 4 \end{array}$$

with $p_1, p_2, p_3, p_4, p_5 \geq 0$

- Constraints 2 and 3 can be combined.
- Variables p_1, p_2 always appear together as $p_1 - p_2$.
- Multiply constraint 1 by -1 to get a positive b:

Let $p_6 = p_1 - p_2$. We now have:

$$
\begin{array}{lrrrrl}
\min w = & 12p_6 & -9p_3 & +10p_4 & -7p_5 & \\
 & -4p_6 & +7p_3 & -8p_4 & +3p_5 & \leq 8 \\
 & -2p_6 & +5p_3 & -5p_4 & +7p_5 & = 5 \\
 & -8p_6 & +6p_3 & -4p_4 & +9p_5 & \geq 4
\end{array}
$$

with p_6 URS, and $p_3, p_4, p_5 \geq 0$.
Since p_6 and p_4 only appear in the constraints with a negative coefficient, convert them to negative constraints.

Here, we'll convert to x: Let $x_1 = -p_6$, $x_2 = p_3$, $x_3 = -p_4$ and $x_4 = p_5$.

The dual is (converting to a MAX):

$$\begin{aligned} \max z = \quad & 12x_1 + 9x_2 + 10x_3 + 7x_4 \\ \text{s.t.} \quad & 4x_1 + 7x_2 + 8x_3 + 3x_4 \le 8 \\ & 2x_1 + 5x_2 + 5x_3 + 7x_4 = 5 \\ & 8x_1 + 6x_2 + 4x_3 + 9x_4 \ge 4 \\ & x_1 \text{ URS}, x_2 \ge 0, x_3 \le 0, x_4 \ge 0 \end{aligned}$$

Summary of the Conversion:

$$\begin{aligned} \max z = \quad & 12x_1 + 9x_2 + 10x_3 + 7x_4 \\ \text{s.t.} \quad & 4x_1 + 7x_2 + 8x_3 + 3x_4 \leq 8 \\ & 2x_1 + 5x_2 + 5x_3 + 7x_4 = 5 \\ & 8x_1 + 6x_2 + 4x_3 + 9x_4 \geq 4 \\ & x_1 \text{ URS}, x_2 \geq 0, x_3 \leq 0, x_4 \geq 0 \end{aligned}$$

$$\begin{aligned} \min w = \quad & 8y_1 + 5y_2 + 4y_3 \\ \text{s.t.} \quad & 4y_1 + 2y_2 + 8y_3 = 12 \\ & 7y_1 + 5y_2 + 6y_3 \geq 9 \\ & 8y_1 + 5y_2 + 4y_3 \leq 10 \\ & 3y_1 + 7y_2 + 9y_3 \geq 7 \\ & y_1 \geq 0, y_2 \text{ URS}, y_3 \leq 0 \end{aligned}$$

Summary:

Primal:	Dual:	
max	min	normal
$\leq$ constraint	≥ 0 variable	normal
$\geq$ constraint	≤ 0 variable	
Equality constraint	URS variable	
≥ 0 variable	$\geq$ constraint	normal
≤ 0 variable	$\leq$ constraint	
URS variable	Equality constraint	

Using a table (*=Not "normal")

$$\begin{aligned} \min w = \quad & 8y_1 + 5y_2 + 4y_3 \\ \text{s.t.} \quad & 4y_1 + 2y_2 + 8y_3 = 12 \\ & 7y_1 + 5y_2 + 6y_3 \geq 9 \\ & 8y_1 + 5y_2 + 4y_3 \leq 10 \\ & 3y_1 + 7y_2 + 9y_3 \geq 7 \end{aligned}$$

$$y_1 \geq 0, y_2 \text{ URS}, y_3 \leq 0$$

	x_1?	x_2?	x_3?	x_4?	
$y_1 \geq 0$	4	7	8	3	?8
y_2 urs(∗)	2	5	5	7	?5
$y_3 \leq 0$(∗)	8	6	4	9	?4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

Using a table

	x_1?	x_2?	x_3?	x_4?	
$y_1 \geq 0$	4	7	8	3	≤ 8
y_2 urs(∗)	2	5	5	7	?5
$y_3 \leq 0(*)$	8	6	4	9	?4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

Using a table

	x_1?	x_2?	x_3?	x_4?	
$y_1 \geq 0$	4	7	8	3	≤ 8
y_2 urs$(*)$	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	?4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

Using a table

	x_1?	x_2?	x_3?	x_4?	
$y_1 \geq 0$	4	7	8	3	≤ 8
y_2 urs$(*)$	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	≥ 4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

Using a table

	x_1 URS	x_2?	x_3?	x_4?	
$y_1 \geq 0$	4	7	8	3	≤ 8
y_2 urs$(*)$	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	≥ 4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

Using a table

	x_1 URS	$x_2 \geq 0$	x_3?	x_4?	
$y_1 \geq 0$	4	7	8	3	≤ 8
y_2 urs$(*)$	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	≥ 4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

Using a table

	x_1 URS	$x_2 \geq 0$	$x_3 \leq 0$	x_4?	
$y_1 \geq 0$	4	7	8	3	≤ 8
y_2 urs$(*)$	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	≥ 4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

Using a table

	x_1 URS	$x_2 \geq 0$	$x_3 \leq 0$	$x_4 \geq 0$	
$y_1 \geq 0$	4	7	8	3	≤ 8
y_2 urs$(*)$	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	≥ 4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

Using a table- Here are Primal and Dual

	x_1 URS	$x_2 \geq 0$	$x_3 \leq 0$	$x_4 \geq 0$	
$y_1 \geq 0$	4	7	8	3	≤ 8
y_2 urs$(*)$	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	≥ 4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

$$\max z = 12x_1 + 9x_2 + 10x_3 + 7x_4$$

$$\text{s.t.} \quad \begin{aligned} 4x_1 + 7x_2 + 8x_3 + 3x_4 &\leq 8 \\ 2x_1 + 5x_2 + 5x_3 + 7x_4 &= 5 \\ 8x_1 + 6x_2 + 4x_3 + 9x_4 &\geq 4 \end{aligned}$$

$$x_1 \text{ URS}, x_2 \geq 0, x_3 \leq 0, x_4 \geq 0$$

$$\min w = 8y_1 + 5y_2 + 4y_3$$

$$\text{s.t.} \quad \begin{aligned} 4y_1 + 2y_2 + 8y_3 &= 12 \\ 7y_1 + 5y_2 + 6y_3 &\geq 9 \\ 8y_1 + 5y_2 + 4y_3 &\leq 10 \\ 3y_1 + 7y_2 + 9y_3 &\geq 7 \end{aligned}$$

$$y_1 \geq 0, y_2 \text{ URS}, y_3 \leq 0$$

Example 2

Use a table to write the dual:

$$\begin{aligned} \max z = \quad & 2x_1 + x_2 \\ \text{st} \quad & x_1 + x_2 = 2 \\ & 2x_1 - x_2 \geq 3 \\ & x_1 - x_2 \leq 1 \\ & x_1 \geq 0, x_2 \text{ urs} \end{aligned}$$

	$x_1 \geq 0$	x_2 urs$(*)$	
y_1 ?	1	1	$= 2(*)$
y_2 ?	2	-1	$\geq 3(*)$
y_3 ?	1	-1	≤ 1
	? 2	? 1	

	$x_1 \geq 0$	x_2 urs$(*)$	
y_1 urs	1	1	$= 2(*)$
$y_2 \leq 0$	2	-1	$\geq 3(*)$
$y_3 \geq 0$	1	-1	≤ 1
	≥ 2	$= 1$	

Use a table to write the dual:

$$\begin{aligned} \max z = \quad & 2x_1 + x_2 \\ \text{st} \quad & x_1 + x_2 = 2 \\ & 2x_1 - x_2 \geq 3 \\ & x_1 - x_2 \leq 1 \\ & x_1 \geq 0, x_2 \text{ urs} \end{aligned}$$

	$x_1 \geq 0$	x_2 urs(∗)	
y_1 urs	1	1	$= 2(*)$
$y_2 \leq 0$	2	-1	$\geq 3(*)$
$y_3 \geq 0$	1	-1	≤ 1
	≥ 2	$= 1$	

$$\begin{aligned} \min w = \quad & 2y_1 + 3y_2 + y_3 \\ \text{st} \quad & y_1 + 2y_2 + y_3 \geq 2 \\ & y_1 - y_2 - y_3 = 1 \\ & y_1 \text{ urs}, y_2 \leq 0, y_3 \geq 0 \end{aligned}$$