Summary of 6.7

Theorems

Lemma 1: Weak Duality

If $\mathbf{x}$ is any feasible point for the primal, and $\mathbf{y}$ is any feasible point for the dual, then $z \le w$.

This line is important for the proof, and also gives some insight into what's going on:

$$\mathbf{x}^T\mathbf{c} \le \mathbf{y}^T A\mathbf{x} \le \mathbf{y}^T\mathbf{b}$$

Lemma 2: Strong Duality

Let $\mathbf{x}$, $\mathbf{y}$ be any feasible points to the primal and dual, respectively, so that

$$\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$$

Then, the solutions are optimal for their respective LPs.

Theorem: The Dual Theorem

Let $\mathcal{B}$ be an optimal basis for the primal. Then

$$\mathbf{y} = (\mathbf{c}_{\mathcal{B}}^T B^{-1})^T$$

is an optimal solution to the dual. Furthermore, the optimal values of the primal and dual are equal ($z = w$).

How to Compute the Dual From Optimal Row 0

- Row 0 in the optimal tableau (of a max problem) is given by

$$-\mathbf{c}^T + \mathbf{c}_B^T B^{-1} A$$

- We talked about what the optimal Row 0 coefficients are in each of the following cases.
  - For a slack variable, s_i, the Row 0 coefficient is $\mathbf{c}_B^T(B^{-1})_i = y_1$.
  - For an excess variable, e_i, the Row 0 coefficient is $-\mathbf{c}_B^T(B^{-1})_i = -y_i$.
  - For an artificial variable a_i, the Row 0 coefficient (using big M) is $\mathbf{c}_B^T(B^{-1})_i + M = y_i - M$

Example

Consider the following LP and its dual:

$$\begin{array}{rl} \max & -2x_1 - x_2 + x_3 \\ \text{st} & x_1 + x_2 + x_3 \le 3 \\ & \quad x_2 + x_3 \ge 2 \\ & x_1 + x_3 = 1 \\ & \mathbf{x} \ge 0 \end{array} \qquad \begin{array}{rl} \min w = & 3y_1 + 2y_2 + y_3 \\ \text{s.t.} & y_1 + y_3 \ge -2 \\ & y_1 + y_2 \ge -1 \\ & y_1 + y_2 + y_3 \ge 1 \\ & y_1 \ge 0,\ y_2 \le 0,\ y_3 \text{ URS} \end{array}$$

Here is the optimal tableau. You may assume that s_1 is the slack variable for constraint 1, e_1, a_1 are the excess and artificial variables for constraint 2, and a_2 is the artificial variable for constraint 3.

Find the solution to the dual:

x_1	x_2	x_3	s_1	e_1	a_1	a_2	rhs
4	0	0	0	1	$-1+M$	$2+M$	0
1	0	0	1	1	-1	0	1
-1	1	0	0	-1	1	-1	1
1	0	1	0	0	0	1	1

SOLUTION: If s_1 is the first slack variable, it started with column $[1, 0, 0]$. Therefore, we can use the optimal Row 0 value for y_1. In this case, $y_1 = 0$.

For y_2, we can use either e_1 (the excess variable for the second constraint), or a_1 (the artificial variable for the second constraint:

- Using e_1, the value of $y_2 = -1$ (multiply the Row 0 value by -1).
- Using a_1, the value of $y_2 = (-1 + M) - M = -1$ (subtract M from the optimal Row 0 value).

For y_3, we must use a_2, and so we subtract M from it to get $y_3 = 2$.

In summary, the solution to the dual is (in vector form) $[0, -1, 2]$, and the excess variables will be given by $[4, 0, 0]$ (the numbers in Row 0 corresponding to the original variables).

A Note About Excess Variables

If the original problem is given by $\max \mathbf{c}^T\mathbf{x}$ such that $A\mathbf{x} \leq \mathbf{b}$, then we will introduce slack variables so that the tableau becomes:

$\mathbf{x}$	s_1	s_2	$\cdots$	s_m	rhs
$-\mathbf{c}^T$	0	0	$\cdots$	0	0
A			$I_{m\times m}$		$\mathbf{b}$

Going to the optimal tableau, we get:

$\mathbf{x}$	s_1	s_2	$\cdots$	s_m	rhs
$-\mathbf{c}^T + \mathbf{c}_B^T B^{-1} A$	y_1	y_2	$\cdots$	y_m	$\mathbf{c}_B^T B^{-1}\mathbf{b}$
$B^{-1}A$			B^{-1}		$B^{-1}\mathbf{b}$

where

$$\mathbf{y}^T = \mathbf{c}_B^T B^{-1}$$

as proven in the Dual Theorem. Now, suppose we define the vector $\vec{E}$ as the vector of n excess variables for the dual- $\vec{E} = [e_1, e_2, \ldots, e_n]^T$. Then, in the dual we have the following (substitute in the value for $\mathbf{y}$):

$$A^T\mathbf{y} = \mathbf{c} + \vec{E} \quad \Rightarrow \quad \vec{E}^T = -\mathbf{c}^T + \mathbf{c}_B^T B^{-1} A$$