

Review Questions, Final Exam

A few general questions

1. What does the Representation Theorem say (in linear programming)?

In words, the representation theorem says that any feasible point can be written as a sum of a convex combination of BFS with (optionally) a direction of unboundedness. In symbols, if $\mathbf{b}_1, \dots, \mathbf{b}_k$ are the BFS and \mathbf{d} is a direction of unboundedness, then:

$$\mathbf{x} = \sum_{n=1}^k \sigma_n \mathbf{b}_n + \mathbf{d}$$

where $\sigma_n \geq 0$ and $\sum_n \sigma_n = 1$.

2. What is the Fundamental Theorem of Linear Programming?

The Fundamental Theorem states that, if an LP has an optimal solution, then it will occur at a BFS.

Note that it is possible that optimal solutions may exist that are not BFS, for example, if two BFS give optimal solutions, then the line segment in between does as well. The theorem is fundamental in the sense that it tells us to look at the set of BFS for an optimal solution.

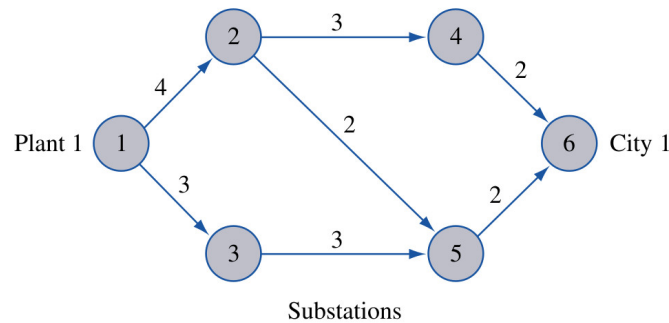
3. What is the main idea behind the Simplex Method? (Think about what it is doing graphically- How does the algorithm start, how does it proceed?)

The Simplex Method begins with a BFS, then it proceeds to check adjacent BFS, where the value of z is increasing (it can also stay the same).

From Chapter 8

The review for this chapter was pretty terrible, so here are some review questions.

- Write the shortest path problem as (i) a transshipment problem, and (ii) a linear program. For specificity, use the PowerCo network below (Figure 2, p 414). (Hints: For transshipment, we have one supply, one demand, and a bunch of warehouses. For the LP, you could write it from the transshipment problem.). Finally, find the shortest path from Plant 1 to City 1 using Dijkstra's algorithm.



SOLUTION:

- As a transshipment problem,

	2	3	4	5	6	
Plant 1	4	3	M	M	M	1
2	0	M	3	2	M	1
3	M	0	M	3	M	1
4	M	M	0	M	2	1
5	M	M	M	0	2	1
	1	1	1	1	1	

- For the linear program, we have:

$$\min 4x_{12} + 3x_{13} + 3x_{24} + 2x_{25} + 3x_{35} + 2x_{46} + 2x_{56}$$

such that (sums across first, then sums down each column):

$$\begin{array}{rcl}
 x_{12} + x_{13} & = & 1 \\
 x_{22} + x_{24} + x_{25} & = & 1 \\
 x_{33} + x_{35} & = & 1 \\
 x_{44} + x_{46} & = & 1 \\
 x_{55} + x_{56} & = & 1 \\
 \hline
 x_{12} + x_{22} & = & 1 \\
 x_{13} + x_{33} & = & 1 \\
 x_{24} + x_{44} & = & 1 \\
 x_{25} + x_{35} + x_{55} & = & 1 \\
 x_{46} + x_{56} & = & 1
 \end{array}$$

(And all $x_{ij} \geq 0$)

- For the Dijkstra algorithm, we have:

	1	2	3	4	5	6
1	0₁	4 ₁	3 ₁	∞_1	∞_1	∞_1
3		4 ₁	3₁	∞_1	6 ₃	∞_1
2		4₁		7 ₂	6 _{2,3}	∞_1
5				7 ₂	6_{2,3}	∞_1
4				7₂		8 ₅
6						8₅

This tells us that the shortest length to node 6 is 8 units, and we get that by taking either $1 \rightarrow 2 \rightarrow 5 \rightarrow 6$, or $1 \rightarrow 3 \rightarrow 5 \rightarrow 6$.

Side Remark: Notice that the first path tells us to fill in a “1” in the cells to the transportation problem: x_{12} , x_{25} , and x_{56} . However, that’s not a BFS for the transportation problem- To fill out the solution, we would have to include x_{33} and x_{44} .

- Given the figure below (Fig 23 from the text), first write the maximum flow problem as a linear program. (Hint: Think about the constraints on the flow for each edge, then for each vertex). Solve the max-flow problem using Ford-Fulkerson. Be sure to write out the residual graphs. Finally, find a cut giving the minimum capacity to show that your solution is correct.

SOLUTION: Recall that, for a flow to be valid, we have some restrictions that make up the linear program:

- The flow on each edge may not be negative, and may not be greater than the capacity.

- Except for the source and sink, flow in must equal flow out.
- Flow out of the source is equal to the flow into the sink.

Writing these in terms of our restrictions, the LP is:

$$\min z = f_{so1} + f_{so2}$$

$$\text{s.t. } f_{so1} \leq 4$$

$$f_{so2} \leq 6$$

$$f_{13} \leq 6$$

$$f_{14} \leq 2$$

$$f_{21} \leq 4$$

$$f_{24} \leq 4$$

$$f_{3si} \leq 6$$

$$f_{43} \leq 1$$

$$f_{4si} \leq 2$$

$$f_{so1} + f_{21} = f_{13}$$

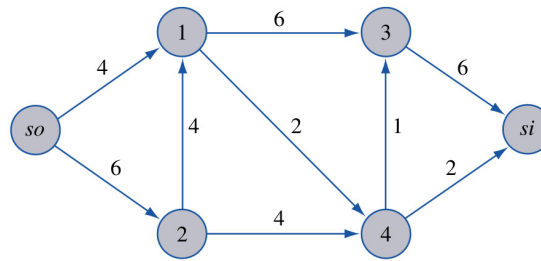
$$f_{so2} = f_{21} + f_{24}$$

$$f_{13} + f_{43} = f_{3si}$$

$$f_{14} + f_{24} = f_{43} + f_{4si}$$

$$f_{so1} + f_{so2} = f_{3si} + f_{4si}$$

FIGURE 23



with all $f_{ij} \geq 0$.

We should find that the maximum flow is 8, and the cut that gives us that:

$$A = \{so, 1, 2, 3, 4\}, B = \{si\}$$

For the details, see the attached handwritten solution for the flow, with the residual graphs.

- Continuing with Figure 23 from the previous question, with the maximum flow, if the cut is:

$$A = \{so, 2, 3\}, B = \{1, 4, si\}$$

then what is the net flow across the cut? What is the capacity of the cut?

SOLUTION: Look at the edges, and determine if they run from A to B or vice versa, or neither:

$A \rightarrow B$	$B \rightarrow A$	Neither
f_{so1}	f_{13}	f_{so2}
f_{21}	f_{43}	f_{14}
f_{3si}		f_{4si}
f_{24}		

For the net flow, we sum f_e from A to B, subtract the flow going the other direction:

$$(4 + 3 + 6 + 3) - (5 + 1) = 10$$

For the capacity of the cut, only sum c_e from A to B:

$$4 + 4 + 6 + 4 = 18$$

4. To use a max-flow for the assignment problem, recall that we have a source node that goes to a node for each person (capacity of 1 each), then each person is attached to a job (capacity of 1 for each edge), and then each job is attached to a sink. (NOTE: We are maximizing the number of compatible pairs.)

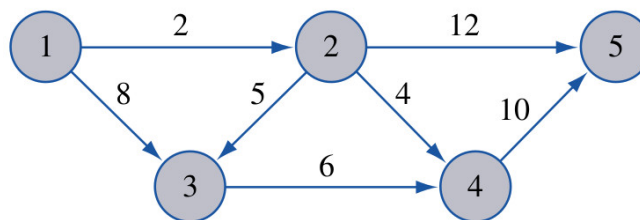
(Problem 7, 8.3) Four workers are available to perform four jobs, but not all workers may be assigned to every job (See the chart below, X marks compatible). Draw the network for the maximum flow problem that can be used to determine whether all jobs can be assigned to a suitable worker.

	Job 1	Job 2	Job 3	Job 4
Worker 1	X	—	—	—
2	X	X	—	—
3	—	X	—	—
4	X	X	X	X

For the details, see the attached handwritten solution.

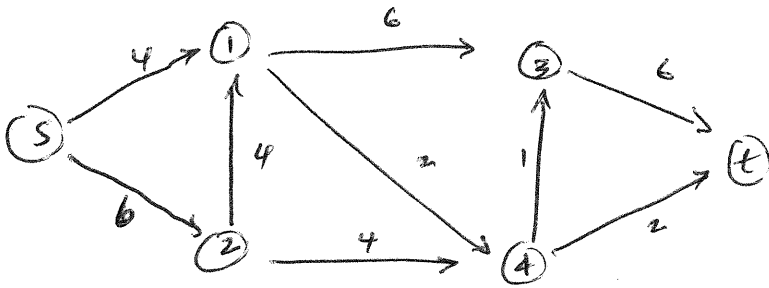
5. (Figure 4 from text below) (a) If we look at the edges as having costs, find the path from node 1 to all other nodes. (b) If we look at the edges as capacities for a flow, find the maximum flow.

FIGURE 4
Network for Problem 2



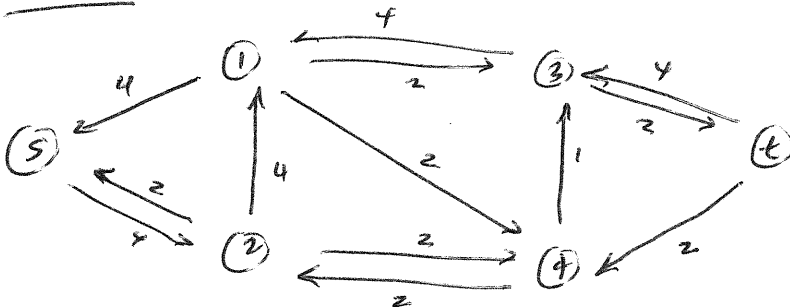
See the attached handwritten solutions.

#2

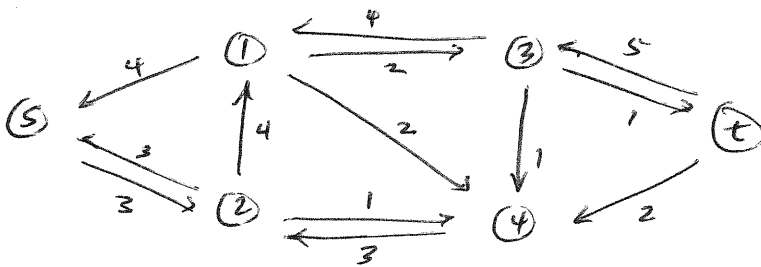


$s-1-3-t$ (1) $\{$ $s-2-4-t$ (2)

Residual:



$s-2-4-3-t$ (1)



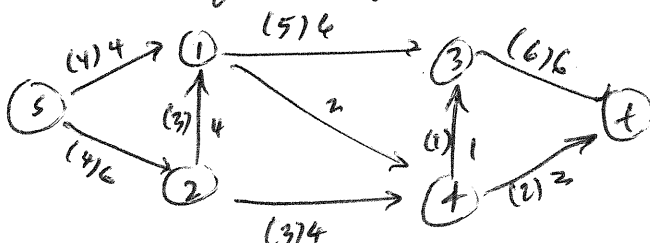
$s-2-1-3-t$ (1) $\text{val}(f) = 8$

This finishes the algorithm.

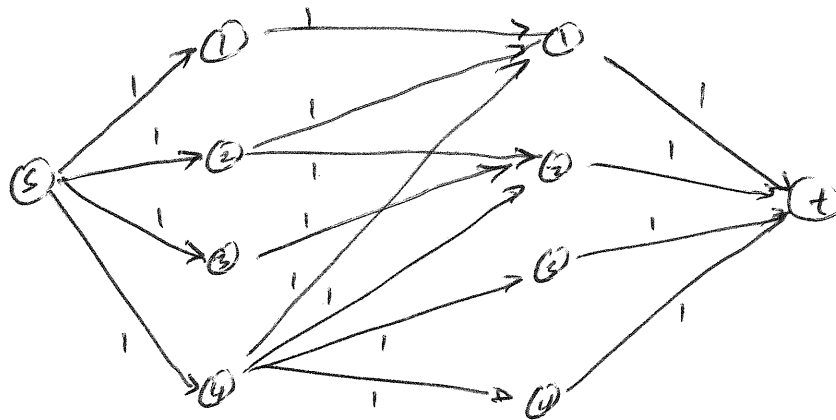
We can cut:

$A = \{s, 1, 2, 3, 4\}$, $B = \{t\}$, cap of cut is 8.

Final value of the flow:



#4 (Prob 7, 8.3)



#5 (a)

	1	2	3	4	5
1	$\boxed{0_1}$	2 ₁	8 ₁	∞_1	∞_1
2		$\boxed{2_1}$	7 ₂	6 ₂	14 ₂
4			7 ₂	$\boxed{6_2}$	14 ₂
3			$\boxed{7_2}$		14 ₂
5					$\boxed{14_2}$

(b) You should get flow of 8.

Cut: $A = \{1, 3\}$, $B = \{2, 4, 5\}$