The Dual

- By Complementary Slackness, if BV x_{ij} is > 0, the corresponding slack in the dual constraint is zero so that the dual constraint is actually equality:

$$u_i + v_j = c_{ij}$$

  For the NBV, $u_i + v_j \geq c_{ij}$. We use the equality to compute u_i, v_j.
- We have an extra variable, so we will set $u_1 = 0$ (This is a random choice).
- Once we have solved for the u_i, v_j in the cells belonging to the BVs, they are used to compute "Row 0" values for the NBV cells.
- If these are all non-negative, the current solution is optimal.

	$v_1 =$	$v_2 =$	$v_3 =$	$v_4 =$	Supply
$u_1 =$	8 35	6	10	9	35
$u_2 =$	9 10	12 20	13 20	7	50
$u_3 =$	14	9	16 10	5 30	40
Demand	45	20	30	30	125

Current Value of $z = 1180$.

	$v_1 =$	$v_2 =$	$v_3 =$	$v_4 =$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 =$	9 10	12 20	13 20	7	50
$u_3 =$	14	9	16 10	5 30	40
Demand	45	20	30	30	125

Current Value of $z = 1180$.

	$v_1 = 8$	$v_2 =$	$v_3 =$	$v_4 =$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 =$	9 10	12 20	13 20	7	50
$u_3 =$	14	9	16 10	5 30	40
Demand	45	20	30	30	125

Current Value of $z = 1180$.

	$v_1 = 8$	$v_2 =$	$v_3 =$	$v_4 =$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 = 1$	9 10	12 20	13 20	7	50
$u_3 =$	14	9	16 10	5 30	40
Demand	45	20	30	30	125

Current Value of $z = 1180$.

	$v_1 = 8$	$v_2 = 11$	$v_3 =$	$v_4 =$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 = 1$	9 10	12 20	13 20	7	50
$u_3 =$	14	9	16 10	5 30	40
Demand	45	20	30	30	125

Current Value of $z = 1180$.

	$v_1 = 8$		$v_2 = 11$		$v_3 = 12$		$v_4 =$		Supply
		8		6		10		9	
$u_1 = 0$	35								35
		9		12		13		7	
$u_2 = 1$	10		20		20				50
		14		9		16		5	
$u_3 =$					10		30		40
Demand	45		20		30		30		125

Current Value of $z = 1180$.

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 =$	Supply
$u_1 = 0$	$\boxed{8}$ 35	$\boxed{6}$	$\boxed{10}$	$\boxed{9}$	35
$u_2 = 1$	$\boxed{9}$ 10	$\boxed{12}$ 20	$\boxed{13}$ 20	$\boxed{7}$	50
$u_3 = 4$	$\boxed{14}$	$\boxed{9}$	$\boxed{16}$ 10	$\boxed{5}$ 30	40
Demand	45	20	30	30	125

Current Value of $z = 1180$.

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 1$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 = 1$	9 10	12 20	13 20	7	50
$u_3 = 4$	14	9	16 10	5 30	40
Demand	45	20	30	30	125

Current Value of $z = 1180$.

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 1$	Supply
$u_1 = 0$	8 / 35	6	10	9	35
$u_2 = 1$	9 / 10	12 / 20	13 / 20	7	50
$u_3 = 4$	14 / (2)	9	16 / 10	5 / 30	40
Demand	45	20	30	30	125

$$c_{ij} - (u_i + v_j) = 14 - (8 + 4) = 14 - 12 = 2$$

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 1$	Supply
$u_1 = 0$	8 35	6 (-5)	10	9	35
$u_2 = 1$	9 10	12 20	13 20	7	50
$u_3 = 4$	14 (2)	9	16 10	5 30	40
Demand	45	20	30	30	125

Current Value of $z = 1180$.

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 1$	Supply
$u_1 = 0$	8 35	6 (-5)	10	9	35
$u_2 = 1$	9 10	12 20	13 20	7	50
$u_3 = 4$	14 (2)	9 (-6)	16 10	5 30	40
Demand	45	20	30	30	125

Current Value of $z = 1180$.

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 1$	Supply
$u_1 = 0$	8 35	6 (-5)	10 (-2)	9	35
$u_2 = 1$	9 10	12 20	13 20	7	50
$u_3 = 4$	14 (2)	9 (-6)	16 10	5 30	40
Demand	45	20	30	30	125

Current Value of $z = 1180$.

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 1$	Supply
$u_1 = 0$	8 35	6 (-5)	10 (-2)	9 (8)	35
$u_2 = 1$	9 10	12 20	13 20	7 (5)	50
$u_3 = 4$	14 (2)	9 (-6)	16 10	5 30	40
Demand	45	20	30	30	125

Current Value of $z = 1180$.

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 1$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 = 1$	9 10	12 20	13 20	7	50
$u_3 = 4$	14	9 ***	16 10	5 30	40
Demand	45	20	30	30	125

Bringing in a new variable creates a loop. In this case,

$$(3,2) - (3,3) - (2,3) - (2,2) \text{ or } 0 - 1 - 2 - 3$$

Increase "even" cells (which are 0, 2), decrease "odd" cells.

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 1$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 = 1$	9 10	12 $20 - \theta$	13 $20 + \theta$	7	50
$u_3 = 4$	14	9 θ	16 $10 - \theta$	5 30	40
Demand	45	20	30	30	125

Increase θ by as much as possible, while maintaining non-negative numbers in the cells. This amount will always be the smallest "odd" cell (in this case, 10).

$$\begin{array}{c|c} 20-\theta & 20+\theta \\ \hline \theta & 10-\theta \end{array} \quad \rightarrow \quad \begin{array}{c|c} 10 & 30 \\ \hline 10 & \end{array}$$

The cell that becomes zero is removed from the set of basic variables.

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 1$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 = 1$	9 10	12 10	13 30	7	50
$u_3 = 4$	14	9 10	16	5 30	40
Demand	45	20	30	30	125

New Value of $z = 1120$

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 1$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 = 1$	9 10	12 10	13 30	7	50
$u_3 = 4$	14	9 10	16	5 30	40
Demand	45	20	30	30	125

New Value of $z = 1120$

The shadow price was -6, and in that cell we added 10 units. This dropped the value of z by 60.

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = ??$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 = 1$	9 10	12 10	13 30	7	50
$u_3 = ??$	14	9 10	16	5 30	40
Demand	45	20	30	30	125

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = ??$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 = 1$	9 10	12 10	13 30	7	50
$u_3 = -2$	14	9 10	16	5 30	40
Demand	45	20	30	30	125

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 7$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 = 1$	9 10	12 10	13 30	7	50
$u_3 = -2$	14	9 10	16	5 30	40
Demand	45	20	30	30	125

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 7$	Supply
	8	6	10	9	
$u_1 = 0$	35	(-5)	(-2)	(2)	35
	9	12	13	7	
$u_2 = 1$	10	10	30	(-1)	50
	14	9	16	5	
$u_3 = -2$	(8)	10	(6)	30	40
Demand	45	20	30	30	125

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 1$	Supply
$u_1 = 0$	8 $35 - \theta$	6 θ	10	9	35
$u_2 = 1$	9 $10 + \theta$	12 $10 - \theta$	13 30	7	50
$u_3 = 4$	14	9 10	16	5 30	40
Demand	45	20	30	30	125

We bring x_{12} into the set of BVs, and by using the loop, we'll remove x_{22} from the set.

	$v_1 = 8$		$v_2 = 11$		$v_3 = 12$		$v_4 = 1$		Supply
		8		6		10		9	
$u_1 = 0$	25		10						35
		9		12		13		7	
$u_2 = 1$	20				30				50
		14		9		16		5	
$u_3 = 4$			10				30		40
Demand	45		20		30		30		125

New value of $z = 1070$. Is it optimal?

	$v_1 = 8$	$v_2 = 6$	$v_3 = 12$	$v_4 = 2$	Supply
$u_1 = 0$	8 25	6 10	10	9	35
$u_2 = 1$	9 20	12	13 30	7	50
$u_3 = 3$	14	9 10	16	5 30	40
Demand	45	20	30	30	125

Recalculating the dual...

	$v_1 = 8$	$v_2 = 6$	$v_3 = 12$	$v_4 = 2$	Supply
$u_1 = 0$	8 25	6 10	10 (-2)	9 (7)	35
$u_2 = 1$	9 20	12 (5)	13 30	7 (4)	50
$u_3 = 3$	14 (3)	9 10	16 (1)	5 30	40
Demand	45	20	30	30	125

Recalculating Row 0...

	$v_1 = 8$	$v_2 = 6$	$v_3 = 12$	$v_4 = 2$	Supply
$u_1 = 0$	8 $25 - \theta$	6 10	10 θ	9	35
$u_2 = 1$	9 $20 + \theta$	12	13 $30 - \theta$	7	50
$u_3 = 3$	14	9 10	16	5 30	40
Demand	45	20	30	30	125

Bring in x_{13}, find the loop. We will be taking out x_{11}.

	$v_1 = 8$	$v_2 = 6$	$v_3 = 12$	$v_4 = 2$	Supply
$u_1 = 0$	8	6 10	10 25	9	35
$u_2 = 1$	9 45	12	13 5	7	50
$u_3 = 3$	14	9 10	16	5 30	40
Demand	45	20	30	30	125

New value of $z = 1020$.

	$v_1 = 6$	$v_2 = 6$	$v_3 = 10$	$v_4 = 2$	Supply
$u_1 = 0$	8 (2)	6 10	10 25	9 (7)	35
$u_2 = 3$	9 45	12 (3)	13 5	7 (2)	50
$u_3 = 3$	14 (5)	9 10	16 (3)	5 30	40
Demand	45	20	30	30	125

Recalculate the dual to see if we are optimal yet... And we are.

	$v_1 = 6$	$v_2 = 6$	$v_3 = 10$	$v_4 = 2$	Supply
$u_1 = 0$	8 (2)	6 10	10 25	9 (7)	35
$u_2 = 3$	9 45	12 (3)	13 5	7 (2)	50
$u_3 = 3$	14 (5)	9 10	16 (3)	5 30	40
Demand	45	20	30	30	125

Recalculate the dual to see if we are optimal yet... And we are. Entering any new variable will increase z.

In Class Example

Given the following tableau with BFS, compute the solution to the dual and determine if it is optimal. If not, say which cell should come into the basis.

	$v_1 =$	$v_2 =$	$v_3 =$	$v_4 =$	Supply
$u_1 =$	cost 2; 5	cost 3	cost 5	cost 6	5
$u_2 =$	cost 2; 7	cost 1; 3	cost 3	cost 5	10
$u_3 =$	cost 3	cost 8; 5	cost 4; 4	cost 6; 6	15
Demand	12	8	4	6	30

In Class Example

	$v_1 = 2$	$v_2 = 1$	$v_3 = -3$	$v_4 = -1$	Supply
$u_1 = 0$	2 5	3 (2)	5 (8)	6 (7)	5
$u_2 = 0$	2 7	1 3	3 (6)	5 (6)	10
$u_3 = 7$	3 (-6)	8 5	4 4	6 6	15
Demand	12	8	4	6	30

In Class Example

	$v_1 = 2$	$v_2 = 1$	$v_3 = -3$	$v_4 = -1$	Supply
$u_1 = 0$	2 5	3 (2)	5 (8)	6 (7)	5
$u_2 = 0$	2 7	1 3	3 (6)	5 (6)	10
$u_3 = 7$	3 (-6)	8 5	4 4	6 6	15
Demand	12	8	4	6	30

Bring in the $(3, 1)$ cell.

$$\begin{array}{c|c} 7-\theta & 3+\theta \\ \hline \theta & 5-\theta \end{array} \quad \rightarrow \quad \begin{array}{c|c} 2 & 8 \\ \hline 5 & \end{array}$$

Now enter these variables, re-compute the dual and the Row 0 values.

	$v_1 = 2$	$v_2 = 1$	$v_3 = 3$	$v_4 = 5$	Supply
$u_1 = 0$	2 5	3 (2)	5 (2)	6 (2)	5
$u_2 = 0$	2 2	1 8	3 (0)	5 (0)	10
$u_3 = 1$	3 5	8 (6)	4 4	6 6	15
Demand	12	8	4	6	30

	$v_1 = 2$	$v_2 = 1$	$v_3 = 3$	$v_4 = 5$	Supply
$u_1 = 0$	2 5	3 (2)	5 (2)	6 (2)	5
$u_2 = 0$	2 2	1 8	3 (0)	5 (0)	10
$u_3 = 1$	3 5	8 (6)	4 4	6 6	15
Demand	12	8	4	6	30

This is optimal.

Degeneracy

	$v_1 =$	$v_2 =$	$v_3 =$	$v_4 =$	Supply
$u_1 =$	4 20	6 20	8	8	40
$u_2 =$	6	8 10	6 50	7	60
$u_3 =$	5	7	6	8 50	50
Demand	20	30	50	50	150

Degeneracy

	$v_1 =$	$v_2 =$	$v_3 =$	$v_4 =$	Supply
$u_1 =$	4 20	6 20	8	8	40
$u_2 =$	6	8 10	6 50	7	60
$u_3 =$	5	7	6	8 50	50
Demand	20	30	50	50	150

We cannot solve for the dual...

Degeneracy

	$v_1 =$	$v_2 =$	$v_3 =$	$v_4 =$	Supply
$u_1 =$	4 20	6 20	8	8	40
$u_2 =$	6	8 10	6 50	7	60
$u_3 =$	5	7	6	8 50	50
Demand	20	30	50	50	150

We cannot solve for the dual...

We must decide on which variable will be basic. (Put ϵ in that cell)

Degeneracy

	$v_1 =$	$v_2 =$	$v_3 =$	$v_4 =$	Supply
$u_1 =$	4 20	6 20	8	8	40
$u_2 =$	6	8 10	6 50	7	60
$u_3 =$	5	7	6	8 50	50
Demand	20	30	50	50	150

We cannot solve for the dual...
We must decide on which variable will be basic. (Put ϵ in that cell)
We do not want a loop!

Degeneracy

	$v_1 =$	$v_2 =$	$v_3 =$	$v_4 =$	Supply
$u_1 =$	4 20	6 20	8 No	8	40
$u_2 =$	6 No	8 10	6 50	7	60
$u_3 =$	5	7	6	8 50	50
Demand	20	30	50	50	150

Degeneracy

	$v_1 =$	$v_2 =$	$v_3 =$	$v_4 =$	Supply
$u_1 =$	4 20	6 20	8	8	40
$u_2 =$	6	8 10	6 50	7	60
$u_3 =$	5	7	6 ϵ	8 50	50
Demand	20	30	50	50	150

Degeneracy

Now we can fill in the dual:

	$v_1 = 4$	$v_2 = 6$	$v_3 = 4$	$v_4 = 6$	Supply
$u_1 = 0$	4 20	6 20	8	8	40
$u_2 = 2$	6	8 10	6 50	7	60
$u_3 = 2$	5	7	6 ϵ	8 50	50
Demand	20	30	50	50	150

Degeneracy

Check for optimality:

	$v_1 = 4$	$v_2 = 6$	$v_3 = 4$	$v_4 = 6$	Supply
$u_1 = 0$	4 20	6 20	8 (4)	8 (2)	40
$u_2 = 2$	6 (0)	8 10	6 50	7 (-1)	60
$u_3 = 2$	5 (-1)	7 (-1)	6 ϵ	8 50	50
Demand	20	30	50	50	150

Degeneracy

We'll choose to bring in x_{32}, which gives us the loop:

	$v_1 = 4$	$v_2 = 6$	$v_3 = 4$	$v_4 = 6$	Supply
$u_1 = 0$	4 20	6 20	8 (4)	8 (2)	40
$u_2 = 2$	6 (0)	8 10	6 50	7 (-1)	60
$u_3 = 2$	5 (-1)	7 (-1)	6 ϵ	8 50	50
Demand	20	30	50	50	150

Degeneracy

$$
\begin{array}{r|c}
10-\theta & 50+\theta \\
\hline
\theta & \epsilon-\theta
\end{array}
\quad \Rightarrow \quad
\begin{array}{r|c}
10 & 50 \\
\hline
\epsilon &
\end{array}
$$

Degeneracy

$$\begin{array}{r|c} 10-\theta & 50+\theta \\ \hline \theta & \epsilon-\theta \end{array} \quad \Rightarrow \quad \begin{array}{c|c} 10 & 50 \\ \hline \epsilon & \end{array}$$

This is a common occurrence, and the reason we use ϵ and not 0.

Degeneracy

$$\begin{array}{r|c} 10-\theta & 50+\theta \\ \hline \theta & \epsilon-\theta \end{array} \quad \Rightarrow \quad \begin{array}{c|c} 10 & 50 \\ \hline \epsilon & \end{array}$$

This is a common occurrence, and the reason we use ϵ and not 0.
This could change our computation of the dual...

Degeneracy, continued

Here are our new values for the dual...

Degeneracy, continued

Here are our new values for the dual...

	$v_1 = 4$	$v_2 = 6$	$v_3 = 4$	$v_4 = 7$	Supply
$u_1 = 0$	4 20	6 20	8 (4)	8 (1)	40
$u_2 = 2$	6 (0)	8 10	6 50	7 (-2)	60
$u_3 = 1$	5 (0)	7 ϵ	6 (1)	8 50	50
Demand	20	30	50	50	150

Degeneracy, continued

Bring x_{24} into the basis, and we have the loop below:

	$v_1 = 4$	$v_2 = 6$	$v_3 = 4$	$v_4 = 7$	Supply
$u_1 = 0$	4 20	6 20	8	8	40
$u_2 = 2$	6	8 $10 - \theta$	6 50	7 θ	60
$u_3 = 1$	5	7 $\epsilon + \theta$	6	8 $50 - \theta$	50
Demand	20	30	50	50	150

Degeneracy, continued

Bring x_{24} into the basis, and we have the loop below:

	$v_1 = 4$	$v_2 = 6$	$v_3 = 4$	$v_4 = 7$	Supply
$u_1 = 0$	4 20	6 20	8	8	40
$u_2 = 2$	6	8 $10 - \theta$	6 50	7 θ	60
$u_3 = 1$	5	7 $\epsilon + \theta$	6	8 $50 - \theta$	50
Demand	20	30	50	50	150

With $\theta = 10$, we will remove the degeneracy!

Continuing...

Here is the new tableau with the new dual values computed. We only show negative values of $c_{ij} - (u_i + v_j)$.

	$v_1 = 4$	$v_2 = 6$	$v_3 = 6$	$v_4 = 7$	Supply
$u_1 = 0$	[4] 20	[6] 20	[8]	[8]	40
$u_2 = 0$	[6]	[8]	[6] 50	[7] 10	60
$u_3 = 1$	[5]	[7] 10	[6] (-1)	[8] 40	50
Demand	20	30	50	50	150

Continuing...

Bring in x_{33}, and we have an optimal tableau:

	$v_1 = 4$	$v_2 = 6$	$v_3 = 5$	$v_4 = 6$	Supply
$u_1 = 0$	20 [4]	20 [6]	[8]	[8]	40
$u_2 = 1$	[6]	[8]	10 [6]	50 [7]	60
$u_3 = 1$	[5]	10 [7]	40 [6]	[8]	50
Demand	20	30	50	50	150

Next up: Sensitivity Analysis