Solutions to our Ch 6 Worksheet

1. Write the down vectors/matrices that we typically use in our computations. Namely, $\mathbf{c}, \mathbf{c}_B, B$, and B^{-1}.

$$\mathbf{c} = \begin{bmatrix} 60 \\ 30 \\ 20 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{c}_B = \begin{bmatrix} 0 \\ 20 \\ 60 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 8 \\ 0 & \frac{3}{2} & 4 \\ 0 & \frac{1}{2} & 2 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -1/2 & 3/2 \end{bmatrix}$$

2. Using our vector notation, if $\mathcal{B}$ gives the optimal basis, how do we compute the dual, $\mathbf{y} = (\mathbf{c}_B^T B^{-1})^T$ (the transpose makes it a column).

3. Write down the dual (either as an initial tableau or in "normal form").

   In "normal form",

$$\begin{array}{rl} \min\ w & = 48y_1 + 20y_2 + 8y_3 \\ \text{st} & 8y_1 + 4y_2 + 2y_3 \geq 60 \\ & 6y_1 + 2y_2 + \frac{3}{2}y_3 \geq 30 \\ & y_1 + \frac{3}{2}y_2 + \frac{1}{2}y_3 \geq 20 \\ & \mathbf{y} \geq 0 \end{array}$$

4. Using the optimal Row 0 from the primal, write down the solution to the dual:

$$\mathbf{y} = [0, 10, 10]^T$$

5. In our "normal form", we have $A\mathbf{x} \leq \mathbf{b}$ for the primal and $A^T\mathbf{y} \geq \mathbf{c}$ for the dual.

   - The "slack" for the primal, given $\mathbf{x}$: $\mathbf{b} - A\mathbf{x}$:
     SOLUTION: These are s_1, s_2, s_3, which are already solved for:

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 0 \\ 0 \end{bmatrix}$$

   - The "slack" for the dual, given $\mathbf{y}$: $A^T\mathbf{y} - \mathbf{c}$:
     SOLUTION: These are e_1, e_2, e_3 for the dual, which we can get from (3):

$$\begin{array}{rl} 8(0) + 4(10) + 2(10) - e_1 & = 60 \\ 6(0) + 2(10) + 1.5(10) - e_2 & = 30 \\ (0) + (1.5)(10) + 0.5(10) - e_3 & = 20 \end{array} \quad \Rightarrow \quad \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

   You might notice there's a vector just like this in the final Row 0 of the primal!

6. What is the shadow price for each constraint?

   SOLUTION: The shadow prices are the solutions to the dual, $0, 10, 10$ for constraints 1, 2, and 3, respectively.

7. Write down the inequalit(ies) we need for Δ if we change the coefficient of x_2 from 30 to $30 + \Delta$, and we want the current basis to remain optimal.

   SOLUTION: Since x_2 is a NBV, we can compute this by either: $5 - \Delta > 0$ so $\Delta < 5$, or by using the second constraint of the dual:

$$6(0) + 2(10) + 1.5(10) \geq 30 + \Delta \quad \Rightarrow \quad 5 \geq \Delta$$

8. Write down the inequalit(ies) we need for Δ if we change the coefficient of x_3 from 20 to $20 + \Delta$, and we want the current basis to remain optimal.

   SOLUTION: This is a change to a NBV, so we take:

$$\begin{array}{rllllll} \text{Old Row 0}: & X & 5 & X & X & 10 & 10 \\ +(\Delta)(\text{Row 2}): & X & -2\Delta & X & X & 2\Delta & -4\Delta \\ \hline & 0 & 5-2\Delta & 0 & 0 & 10+2\Delta & 10-4\Delta \end{array} \quad \Rightarrow \quad -5 < \Delta < \frac{5}{2}$$

9. How does changing a *column* of A effect the dual? Use this to see what would happen if we change the column for x_2 (tables) to be $[5, 2, 2]^T$- Is it now worth it to make tables?

   SOLUTION: Changing the 2d column of A means changing the 2d constraint for the dual, so we'll go ahead and check that, using our current solution to the dual:

$$5(0) + 2(10) + 2(10) \geq 30? \quad \Rightarrow \quad \text{Yes.}$$

   Interpretation: The dual is still feasible, so therefore, the current basis for the primal is still optimal. That means we should NOT bring in x_2 (keep x_2 at 0).

10. How does creating a new column of A effect the dual? Use this to see if it makes sense to manufacture footstools, where we sell them for \$15 each, and the resources are $[1, 1, 1]^T$.

   SOLUTION: Adding a new column (or activity) in the primal corresponds to adding a new constraint to the dual:

$$1(0) + 1(10) + 1(10) \geq 15? \quad \Rightarrow \quad \text{Yes.}$$

   Therefore, the dual is still feasible, and the current basis for the primal is still optimal. We should not make any footstools at this price (we see that we would need to price them at at least \$20 each).