Review Solutions, Exam 2, Operations Research

1. Prove the weak duality theorem: For any $\mathbf{x}$ feasible for the primal and $\mathbf{y}$ feasible for the dual, then...

   HINT: Consider the quantity $\mathbf{y}^T A\mathbf{x}$.

   SOLUTION: To use the hint, start with a feasible point in the primal and the dual. Then we know the following two equations are satisfied:

$$A\mathbf{x} \le \mathbf{b} \qquad A^T\mathbf{y} \ge \mathbf{c}$$

   In the first expression, multiply both sides by $\mathbf{y}^T$ (this is OK since we assume $\mathbf{x}, \mathbf{y} \ge 0$). Multiply the second expression by $\mathbf{x}^T$.

$$\mathbf{y}^T A\mathbf{x} \le \mathbf{y}^T\mathbf{b} \qquad \text{and} \qquad \mathbf{x}^T A^T\mathbf{y} \ge \mathbf{x}^T\mathbf{c}$$

   Noting that $\mathbf{x}^T A^T \mathbf{y} = (A\mathbf{x})^T\mathbf{y} = \mathbf{y}^T(A\mathbf{x})$, we get that:

$$\mathbf{c}^T\mathbf{x} \le \mathbf{y}^T A\mathbf{x} \le \mathbf{b}^T\mathbf{y}$$

2. Show that the solution to the dual is $\mathbf{y} = (\mathbf{c}_B^T B^{-1})^T$ (if the primal and dual are both feasible).

   - Show that $\mathbf{y}$ is feasible.
     SOLUTION: $\mathbf{y}$ is feasible if $A^T\mathbf{y} \ge \mathbf{c}$, or, in row form, $-\mathbf{c}^T + \mathbf{y}^T A \ge 0$:

$$-\mathbf{c}^T + \mathbf{y}^T A = -\mathbf{c}^T + (\mathbf{c}_B^T B^{-1})A = -\mathbf{c}^T + \mathbf{c}_B^T B^{-1} A \ge 0$$

     The last inequality is true since $\mathbf{x}$ is optimal for the primal.
   - Show that $z = w$

$$w = \mathbf{y}^T\mathbf{b} = \mathbf{c}_B^T B^{-1}\mathbf{b} = z$$

     since we have an optimal tableau for the primal.

3. Solve using big-M:

$$\begin{array}{rrrrl} \max & z = & 5x_1 & -x_2 & \\ & \text{st} & 2x_1 & +x_2 & = 6 \\ & & x_1 & +x_2 & \le 4 \\ & & & x_2 & \le 3 \\ & & x_1, & x_2 & \ge 0 \end{array}$$

   SOLUTION: We only have one artificial variable (for the equality). Here's the initial tableau- To get a BFS, we need to get rid of the M in Row 0:

x_1	x_2	a_1	s_2	s_3	rhs
-5	1	M	0	0	0
2	1	1	0	0	6
1	1	0	1	0	4
0	1	0	0	1	3

$\rightarrow$

x_1	x_2	a_1	s_2	s_3	rhs
$-5-2M$	$1-M$	0	0	0	$-6M$
2	1	1	0	0	6
1	1	0	1	0	4
0	1	0	0	1	3

Pivot in column 1, row 1:

x_1	x_2	a_1	s_2	s_3	rhs
0	7/2	$\frac{5}{2}+M$	0	0	15
1	1/2	1/2	0	0	3
0	1/2	−1/2	1	0	1
0	1	0	0	1	3

This is our optimal solution: $x_1 = 3, x_2 = 0$, with $z = 15$.

4. Solve the last problem again using the two phase method.

   In two-phase, here is our initial tableau:

x_1	x_2	a_1	s_2	s_3	rhs
0	0	1	0	0	0
2	1	1	0	0	6
1	1	0	1	0	4
0	1	0	0	1	3

$\rightarrow$

x_1	x_2	a_1	s_2	s_3	rhs
−2	−1	0	0	0	−6
2	1	1	0	0	6
1	1	0	1	0	4
0	1	0	0	1	3

   Pivot in the first column, first row:

x_1	x_2	a_1	s_2	s_3	rhs
0	7/2	$\frac{5}{2}+M$	0	0	15
1	1/2	1/2	0	0	3
0	1/2	−1/2	1	0	1
0	1	0	0	1	3

$\rightarrow$

x_1	x_2	a_1	s_2	s_3	rhs
0	0	1	0	0	0
1	1/2	1/2	0	0	3
0	1/2	−1/2	1	0	1
0	1	0	0	1	3

   End of Phase I. Now we have a feasible point, we can bring in our original Row 0 (and remove the artificial variable). We perform the row operations to maintain our columns of the identity.

x_1	x_2	s_2	s_3	rhs
−5	1	0	0	0
1	1/2	0	0	3
0	1/2	1	0	1
0	1	0	1	3

$\rightarrow$

x_1	x_2	s_2	s_3	rhs
0	7/2	0	0	15
1	1/2	0	0	3
0	1/2	1	0	1
0	1	0	1	3

   And this is the optimal tableau, with $z = 15$.

5. Use the simplex algorithm to get a tableau that is suitable for the dual simplex algorithm. In doing so, show that the problem is infeasible, but the dual is feasible.

$$\begin{array}{rl} \min\ z = & -3x_1 + x_2 \\ \text{st} & x_1 - 2x_2 \geq 2 \\ & -x_1 + x_2 \geq 3 \\ & x_1, x_2 \geq 0 \end{array}$$

   SOLUTION: Writing this with columns of the identity in mind, we would get the tableau on the left, below. To get this in the form for the dual simplex, Row 0 needs to be all non-negative. So, pivot in the first column:

x_1	x_2	e_1	e_2	rhs
-3	1	0	0	0
-1	2	1	0	-2
1	-1	0	1	-3

$\Rightarrow$

x_1	x_2	e_1	e_2	rhs
0	-2	0	3	-9
0	1	1	1	-5
1	-1	0	1	-3

We can keep going- Now pivot in column 2:

x_1	x_2	e_1	e_2	rhs
0	0	2	5	-19
0	1	1	1	-5
1	0	1	2	-8

This is now feasible for the **dual**, but not feasible for the primal. However, we cannot pivot anywhere, so the primal remains infeasible.

6. Consider the LP and the optimal tableau with missing Row 0.

$$\begin{aligned} \max z = \; & 3x_1 + x_2 \\ \text{s.t.} \; & 2x_1 + x_2 \le 4 \\ & 3x_1 + 2x_2 \ge 6 \\ & 4x_1 + 2x_2 = 7 \\ & x_1, x_2 \ge 0 \end{aligned}$$

x_1	x_2	s_1	e_2	a_2	a_3	rhs
0	0	1	0	0	$-\frac{1}{2}$	$\frac{1}{2}$
0	1	0	-2	2	$-\frac{3}{2}$	$\frac{3}{2}$
1	0	0	1	-1	1	1

Find Row 0.

SOLUTION: Row 0 coefficients are given by

$$-\mathbf{c}^T + \mathbf{c}_B^T B^{-1} A \qquad \text{where} \qquad \mathbf{c}^T = [3, 1, 0, 0, -M, -M] \quad \text{and} \quad \mathbf{c}_B^T = [0, 1, 3]$$

The columns in the tableau are already the columns for $B^{-1}A$, so we can use them. Here are the Row zero coefficients:

- For e_2: $[0, 1, 3][0, -2, 1]^T = 1$
- For a_2: $M + [0, 1, 3][0, 2, -1] = -1 + M$
- For a_3: $M + [0, 1, 3][-1/2, -3/2, 1]^T = 3/2 + M$
- Optimal z is $[0, 1, 3][1/2, 3/2, 1]^T = 9/2$
- We have zeros for x_1, x_2, s_1.

Therefore, the optimal tableau is:

x_1	x_2	s_1	e_2	a_2	a_3	rhs
0	0	0	1	$-1 + M$	$3/2 + M$	$9/2$
0	0	1	0	0	$-\frac{1}{2}$	$\frac{1}{2}$
0	1	0	-2	2	$-\frac{3}{2}$	$\frac{3}{2}$
1	0	0	1	-1	1	1

7. Give an argument why, if the primal is unbounded, then the dual must be infeasible.

SOLUTION:

Suppose the dual was feasible. Then there exists a $\mathbf{y}$ that satisfies the constraints for the dual. But in that case, for every $\mathbf{x}$ in the primal,

$$\mathbf{c}^T\mathbf{x} \le \mathbf{b}^T\mathbf{y}$$

But we said that the primal was unbounded, so that $\mathbf{c}^T\mathbf{x} \to \infty$. This is a contradiction- Thus, the dual must be infeasible.

8. In solving the following LP, we obtain the optimal tableau shown:

$$\begin{array}{rl} \max z = & 6x_1 + x_2 \\ \text{st} & x_1 + x_2 \le 5 \\ & 2x_1 + x_2 \le 6 \\ & x_1, x_2 \ge 0 \end{array} \quad \Rightarrow \quad \begin{array}{cccc|c} x_1 & x_2 & s_1 & s_2 & rhs \\ 0 & 2 & 0 & 3 & 18 \\ \hline 0 & 1/2 & 1 & -1/2 & 2 \\ 1 & 1/2 & 0 & 1/2 & 3 \end{array}$$

(a) If we add a new constraint, is it possible that we can increase z? Why or why not?

SOLUTION: Since the feasible set *decreases* (that is, the new feasible set is a subset of the original feasible set), then the value of z must either decrease or stay the same, since the current solution is over a larger set.

(b) If we add the constraint $3x_1 + x_2 \le 10$, is the current basis still optimal?

SOLUTION: If we add an extra constraint, the set of feasible points is *reduced.* Therefore, if the optimal solution satisfies this constraint, then we're done (we're not adding new points).

In this case, the solution is $x_1 = 3$, $x_2 = 0$ and the constraint is satisfied (and the current solution remains optimal).

(c) If we add the constraint $x_1 - x_2 \ge 6$, we can quickly see that the optimal solution changes. Find out if we have a new optimal solution or if we have made the problem infeasible.

SOLUTION: If the optimal solution does not satisfy the constraint (as in this case), then we either have a new optimal solution or no feasible points. Let's add in this constraint into the tableau and see if it can be incorporated into it.

Adding the constraint adds a new row and column:

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & e_3 & rhs \\ 0 & 2 & 0 & 3 & 0 & 18 \\ \hline 0 & 1/2 & 1 & -1/2 & 0 & 2 \\ 1 & 1/2 & 0 & 1/2 & 0 & 3 \\ -1 & 1 & 0 & 0 & 1 & -6 \end{array} \qquad \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & e_3 & rhs \\ 0 & 2 & 0 & 3 & 0 & 18 \\ \hline 0 & 1/2 & 1 & -1/2 & 0 & 2 \\ 1 & 1/2 & 0 & 1/2 & 0 & 3 \\ 0 & 3/2 & 0 & 1/2 & 1 & -3 \end{array}$$

The last constraint can never be satisfied. Infeasible.

(d) Same question as the last one, but let's change the constraint to $8x_1 + x_2 \le 12$.

SOLUTION: In this case, the current optimal solution does not satisfy the constraint (therefore, the current solution is no longer optimal). We need to check whether we have a new optimal solution or if we have made the problem infeasible. First eliminate x_1 as we did before:

x_1	x_2	s_1	s_2	s_3	rhs
0	2	0	3	0	18
0	1/2	1	−1/2	0	2
1	1/2	0	1/2	0	3
8	1	0	0	1	12

$-8R_2 + R_3 \rightarrow R_3$

x_1	x_2	s_1	s_2	s_3	rhs
0	2	0	3	0	18
0	1/2	1	−1/2	0	2
1	1/2	0	1/2	0	3
0	−3	0	−4	1	−12

We pivot using the dual simplex to try to make the solution feasible. Using the ratio test, pivot in the x_2 column:

x_1	x_2	s_1	s_2	s_3	rhs
0	0	0	1/3	2/3	10
0	0	1	−7/6	1/6	0
1	0	0	−1/6	1/6	1
0	1	0	4/3	−1/3	4

And now we're feasible and optimal again! (Notice that z decreased).

(e) If I add a new variable x_3 so that:

$$\begin{array}{rl} \max z = & 6x_1 + x_2 + x_3 \\ \text{st} & x_1 + x_2 + 2x_3 \leq 5 \\ & 2x_1 + x_2 + x_3 \leq 6 \\ & x_1, x_2 \geq 0 \end{array}$$

Does the current basis stay optimal? Answer two ways- One using the optimal tableau, and the second using the dual.

SOLUTION:

- Using the optimal tableau, we "price out" the new variable. The new column will be $B^{-1}\mathbf{a}_3$, or

$$\begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$$

  The new Row 0 value will be $-c_3 + \mathbf{c}_B^T B^{-1}\mathbf{a}_3$. Notice that the order given in the optimal tableau for the basic variables is $\{s_1, x_1\}$, so $\mathbf{c}_B^T = [0, 6]$:

$$-1 + [0, 6] \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} = -1 + 3 = 2 > 0$$

- Using the dual, the addition of the new variable adds a new constraint to the dual:

$$2y_1 + y_2 \geq 1$$

  with the current dual being $y_1 = 0$, $y_2 = 3$. In this case, the new constraint is satisfied, so the dual remains optimal (and so does the primal).

9. Solve the following "mixed constraint" problem using a combination of the simplex and the dual simplex.

$$\begin{array}{rrrrl} \min & z = & -x_1 & +x_2 & \\ & \text{st} & -x_1 & +x_2 & \leq 3 \\ & & & x_2 & \geq 6 \\ & & 2x_1 & +x_2 & \leq 18 \end{array}$$

SOLUTION: Change to a max, then multiply the second row by -1 to get the tableau:

x_1	x_2	s_1	e_2	s_3	
-1	1	0	0	0	0
-1	1	1	0	0	3
0	-1	0	1	0	-6
2	1	0	0	1	18

We use the regular simplex method first- pivot on the "2" in the lower left:

x_1	x_2	s_1	e_2	s_3	
0	3/2	0	0	1/2	9
0	3/2	1	0	1/2	12
0	-1	0	1	0	-6
1	1/2	0	0	1/2	9

Now we perform one iteration of the dual simplex method- We use Row 2, then column x_2:

x_1	x_2	s_1	e_2	s_3	
0	0	0	3/2	1/2	0
0	0	1	3/2	1/2	3
0	1	0	-1	0	6
1	0	0	1/2	1/2	6

We are now optimal: $x_1 = x_2 = 6$, with $z = 0$.

Side Remark: What is the solution to the dual? On the one hand, it looks like $\mathbf{y} = [0, 3/2, 1/2]^T$. We can verify that using w, but be careful- In order for our theory to work out, the problem needed to be in "normal form", so we would actually need to multiply the first and last constraints by -1. In that case, $w = -3y_1 + 6y_2 - 18y_3$, and our y values work.

On the other hand, if we use our shortcuts to build the dual from the mixed constraints, then we get $w = 3y_1 + 6y_2 + 18y_3$, where y_1, y_3 are negative. From the optimal Row 0, we would then need to multiply y_1, y_3 by -1 to get the solution to the dual.

10. The Complementary Slackness Theorem says, among other things, that $s_j y_j = 0$. If $s_j > 0$, why should $y_j = 0$? (Explain in words)

    SOLUTION: In the normal form, if $s_j > 0$, then the j^{th} constraint has unused resources (the slack is what is still available). Therefore, it doesn't matter if we add more resources, since we're not using what we have. The shadow price (and therefore also y_j) is then 0.

11. Be able to prove that each of our "shortcut formulas" for sensitivity analysis works.

- Change in a NBV

  SOLUTION: We'll only prove the formula that doesn't use the dual. In that case, the only change to the initial Row 0 is in $\mathbf{c}$ so that the new Row 0 is:

  $$-(\mathbf{c}^T + \Delta\vec{e}_i^T) + \mathbf{c}_B^T B^{-1} A = -\mathbf{c}^T + \mathbf{c}_B^T B^{-1} A - \Delta\vec{e}_i^T$$

  This means that the only thing that changes is in the i^{th} position, and there we take the old i^{th} coefficient from Row 0 and subtract Δ.

- Change in a BV

  SOLUTION: We'll assume the BV has coordinate i in the initial Row 0, and has coordinate j in $\mathbf{c}_B$. Then the change is given by:

  $$-(\mathbf{c}^T + \Delta\vec{e}_i^T) + (\mathbf{c}_B^T + \Delta\vec{e}_j^T) B^{-1} A$$

  Expanding this,

  $$(-\mathbf{c}^T + \mathbf{c}_B^T B^{-1} A) - \Delta\vec{e}_i^T + \Delta(B^{-1}A)_j$$

  where the first term is Row 0 from the final tableau, and the last term is the j^{th} row of the final tableau. The middle term only effects the BV in the i^{th} position, so if we say that the BVs sum to zero, we can drop the middle term and take the sum for the NBVs.

- Change in RHS

  SOLUTION: The RHS of the table changes from $\mathbf{b}$ to $\mathbf{b} + \Delta\vec{e}_i$ so that the RHS of the final tableau becomes

  $$B^{-1}(\mathbf{b} + \Delta\vec{e}_i) = B^{-1}\mathbf{b} + \Delta(B^{-1})_i$$

  where the subscript denotes the i^{th} column of B^{-1}.

- Change in a column corresponding to a NBV.

  SOLUTION: We could do this by not using the dual. In that case, the new column in the final tableau and the new Row 0 entry of the final tableau would be:

  $$B^{-1}\mathbf{a}_i, \text{ and } \qquad -c_i + \mathbf{c}_B^T B^{-1}\mathbf{a}_i > 0$$

  And we might notice that this last expression is the constraint from dual (except for the "="): $\mathbf{y}^T \mathbf{a}_i \geq c_i$

- Adding a new "activity" (column).

  SOLUTION: Basically the exact same computations as in the previous problem.

12. Consider the LP below:

$$\begin{array}{rrrrrrl}
\max & z = & 2x_1 & +2x_2 & & & \\
& \text{st} & x_1 & & +x_3 & +x_4 & \leq 1 \\
& & & x_2 & +x_3 & -x_4 & \leq 1 \\
& & x_1 & +x_2 & +2x_3 & & \leq 3 \\
\mathbf{x} \geq 0 & & & & & &
\end{array}$$

(a) Write out the dual.

$$\begin{array}{rrrrl} \min\ w = & y_1 & +y_2 & +3y_3 & \\ \text{st} & y_1 & & +y_3 & \geq 2 \\ & & y_2 & +y_3 & \geq 2 \\ & y_1 & +y_2 & +2y_3 & \geq 0 \\ & y_1 & -y_2 & & \geq 0 \\ \mathbf{y} \geq 0 & & & & \end{array}$$

(b) Show that $\mathbf{x}^* = [1,1,0,0]^T$ and $\mathbf{y}^* = [1,1,1]^T$ are feasible for the original and dual problems, respectively.

SOLUTION: With the expectation of needing the values of the slacks, we put $\mathbf{x}^*$ into the primal and $\mathbf{y}^*$ into the constraints of the dual:

x_1	x_2	x_3	x_4	s_1	s_2	s_3
1	1	0	0	0	0	1
e_1	e_2	e_3	e_4	y_1	y_2	y_3
0	0	4	0	1	1	1

(c) Show that for this pair of solutions, if $x_j^* > 0$ then the corresponding slack in the dual is 0.

SOLUTION: Yes- check the table from the previous answer.

(d) Show that $\mathbf{y}^*$ is not an optimal solution to the dual.

SOLUTION: Notice if we compute z, w, we know that something is off, but we don't know if the problem is with the primal or the dual.

If we look at the solution to the first part, we see that the variables x_i satisfy the Complementary Slackness conditions ($x_i e_i = 0$), but that $\mathbf{y}$ does not- The problem therefore, must be with the dual.

Given $s_3 > 0$ from the primal, then $y_3 = 0$. Keeping $e_1 = e_2 = e_4 = 0$, we see that $y_1 = y_2 = 2$, and that would given $z = w = 4$.

Therefore, $\mathbf{y} = [1,1,1]^T$ was not optimal for the dual.

(e) Does this contradict the Complementary Slackness Theorem?

13. Prove or disprove using Complementary Slackness: The point $\mathbf{x} = [0,3,0,0,4]^T$ is an optimal solution to the LP:

$$\begin{array}{rrrrrrl} \max\ z = & 5x_1 & +4x_2 & +8x_3 & +9x_4 & +15x_5 & \\ \text{st} & x_1 & +x_2 & +2x_3 & +x_4 & +2x_5 & \leq 11 \\ & x_1 & -2x_2 & -x_3 & +2x_4 & +3x_5 & \leq 6 \\ \mathbf{x} \geq 0 & & & & & & \end{array}$$

SOLUTION: This is similar to the last one. We can fill in the values of the slack variables as well:

x_1	x_2	x_3	x_4	x_5	s_1	s_2
0	3	0	0	4	0	0
e_1	e_2	e_3	e_4	e_5	y_1	y_2

We know that e_2, e_5 both must be zero:

$$\begin{array}{rl} y_1 - 2y_2 & = 4 \\ 2y_1 + 3y_2 & = 15 \end{array} \quad \Rightarrow \quad y_1 = 6, y_2 = 1$$

And checking the other dual conditions to get the values of e_i:

$$e_1 = 2 \quad e_2 = 0 \quad e_3 = 3 \quad e_4 = -1 \qquad e_5 = 0$$

So this $\mathbf{y}$ (derived from the solution to the primal) is infeasible. Thus, $\mathbf{x}$ from the primal is NOT the optimal solution.

Chapter 6 Review Problems:

Solutions are written up separately.