Review Material, Exam 2, Ops Research

1. Prove the weak duality theorem: For any $\mathbf{x}$ feasible for the primal and $\mathbf{y}$ feasible for the dual, then...
   HINT: Put the primal and dual in normal form. Consider the quantity $\mathbf{y}^T A\mathbf{x}$, given the constraints.

2. Show that the solution to the dual is $\mathbf{y} = (\mathbf{c}_B^T B^{-1})^T$ (if the primal and dual are both feasible) by following the steps below. You may assume that the primal and dual are in normal form.
   - Show that $\mathbf{y}$ is feasible for the dual.
   - Show that $z = w$ if we use this formula for $\mathbf{y}$.

3. Solve using big-M:

$$\begin{array}{rrrl} \max\ z = & 5x_1 & -x_2 & \\ \text{st} & 2x_1 & +x_2 & = 6 \\ & x_1 & +x_2 & \leq 4 \\ & & x_2 & \leq 3 \\ & x_1, & x_2 & \geq 0 \end{array}$$

4. Solve the last problem (3) again using the two phase method.

5. Use the simplex algorithm to get a tableau that is suitable for the dual simplex algorithm. In doing so, show that the problem is infeasible, but the dual is feasible.

$$\begin{array}{rl} \min\ z = & -3x_1 + x_2 \\ \text{st} & x_1 - 2x_2 \geq 2 \\ & -x_1 + x_2 \geq 3 \\ & x_1, x_2 \geq 0 \end{array}$$

6. Consider the LP and the optimal tableau with missing Row 0 and missing optimal RHS (assume big-M).

$$\begin{array}{rl} \max z = & 3x_1 + x_2 \\ \text{s.t.} & 2x_1 + x_2 \leq 4 \\ & 3x_1 + 2x_2 \geq 6 \\ & 4x_1 + 2x_2 = 7 \\ & x_1, x_2 \geq 0 \end{array}$$

x_1	x_2	s_1	e_2	a_2	a_3	rhs
0	0	1	0	0	$-1/2$	
0	1	0	-2	2	$-3/2$	
1	0	0	1	-1	1	

   Find Row 0 and the RHS for the optimal tableau (without performing row reductions!)

7. Give an argument why, if the primal is unbounded, then the dual must be infeasible.

8. In solving the following LP, we obtain the optimal tableau shown:

$$\begin{array}{rl} \max z = & 6x_1 + x_2 \\ \text{st} & x_1 + x_2 \leq 5 \\ & 2x_1 + x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{array} \quad \Rightarrow$$

x_1	x_2	s_1	s_2	rhs
0	2	0	3	18
0	$1/2$	1	$-1/2$	2
1	$1/2$	0	$1/2$	3

   (a) If we add a new constraint, is it possible that we can increase z? Why or why not?
   (b) If we add the constraint $3x_1 + x_2 \leq 10$, is the current basis still optimal?
   (c) If we add the constraint $x_1 - x_2 \geq 6$, we can quickly see that the optimal solution changes. Find out if we have a new optimal solution or if we have made the problem infeasible.
   (d) Same question as the last one, but let's change the constraint to $8x_1 + x_2 \leq 12$.
   (e) If I add a new variable x_3 so that:

$$\begin{array}{rl} \max z = & 6x_1 + x_2 + x_3 \\ \text{st} & x_1 + x_2 + 2x_3 \leq 5 \\ & 2x_1 + x_2 + x_3 \leq 6 \\ & x_1, x_2 \geq 0 \end{array}$$

   Does the current basis stay optimal? Answer two ways- One using the optimal tableau, and the second using the dual.

9. Solve the following "mixed constraint" problem using a combination of the simplex and the dual simplex.

$$\begin{array}{rrrrl} \min & z = & -x_1 & +x_2 & \\ & \text{st} & -x_1 & +x_2 & \leq 3 \\ & & & x_2 & \geq 6 \\ & & 2x_1 & +x_2 & \leq 18 \end{array}$$

10. The Complementary Slackness Theorem says, among other things, that $s_j y_j = 0$. If $s_j > 0$, why should $y_j = 0$? (Explain in words)

11. Be able to prove that each of our "shortcut formulas" for sensitivity analysis works.
    - Change in a NBV
    - Change in a BV
    - Change in RHS
    - Change in a column corresponding to a NBV.
    - Adding a new "activity" (column).

12. Consider the LP below:

$$\begin{array}{rrrrrrl} \max & z = & 2x_1 & +2x_2 & & & \\ & \text{st} & x_1 & & +x_3 & +x_4 & \leq 1 \\ & & & x_2 & +x_3 & -x_4 & \leq 1 \\ & & x_1 & +x_2 & +2x_3 & & \leq 3 \\ \mathbf{x} \geq 0 & & & & & & \end{array}$$

    (a) Write out the dual.
    (b) Show that $\mathbf{x}^* = [1, 1, 0, 0]^T$ and $\mathbf{y}^* = [1, 1, 1]^T$ are feasible for the original and dual problems, respectively.
    (c) Show that for this pair of solutions, if $x_j^* > 0$ then the corresponding slack in the dual is 0.
    (d) Show that $\mathbf{y}^*$ is not an optimal solution to the dual.
    (e) Does this contradict the Complementary Slackness Theorem?

13. Prove or disprove using Complementary Slackness: The point $\mathbf{x} = [0, 3, 0, 0, 4]^T$ is an optimal solution to the LP:

$$\begin{array}{rrrrrrrl} \max & z = & 5x_1 & +4x_2 & +8x_3 & +9x_4 & +15x_5 & \\ & \text{st} & x_1 & +x_2 & +2x_3 & +x_4 & +2x_5 & \leq 11 \\ & & x_1 & -2x_2 & -x_3 & +2x_4 & +3x_5 & \leq 6 \\ \mathbf{x} \geq 0 & & & & & & & \end{array}$$

Questions from the Chapter 6 Review:

3, 4 (except 4(b)), 6, 9, 10, 16, 17, 18, 20, 23 (very much the same as 16), 33, 34.