Some Motivating Examples: Complementary Slackness

Suppose we're given the following primal (max) tableau:

x_1	x_2	s_1	s_2	s_3	rhs
-3	-5	0	0	0	0
1	0	1	0	0	4
0	2	0	1	0	12
3	2	0	0	1	18

The current feasible solution for the primal is given below (in tabular form):

x_1	x_2	s_1	s_2	s_3
0	0	4	12	18

$\Rightarrow$

$$\mathcal{B} = \{s_1, s_2, s_3\} \quad \mathbf{c}_B^T = [0, 0, 0] \quad \mathbf{y}^T = \mathbf{c}_B^T B^{-1}$$

$\Rightarrow$

e_1	e_2	y_1	y_2	y_3
-3	-5	0	0	0

In this case, we see that the primal is feasible, and the corresponding dual is not. You can verify the values of e_1, e_2 by substituting the values of y_1, y_2, y_3 into the dual. Below, we will assume that $\mathcal{B} = \{s_1, s_2, x_1\}$ so that B changes as well. The final tableau is given below, with the corresponding values of the dual:

x_1	x_2	s_1	s_2	s_3	rhs
0	-3	0	0	1	18
0	$-2/3$	1	0	$-1/3$	-2
0	2	0	1	0	12
1	$2/3$	0	0	$1/3$	6

$$\mathcal{B} = \{s_1, s_2, x_1\} \quad \mathbf{c}_B^T = [0, 0, 3] \quad \mathbf{y}^T = \mathbf{c}_B^T B^{-1} = [0, 0, 1]^T$$

x_1	x_2	s_1	s_2	s_3
6	0	-2	12	0

e_1	e_2	y_1	y_2	y_3
0	-3	0	0	1

On the left, the primal is not feasible, and on the right, the dual is not feasible either. In both cases, the current value of $z = w = 18$. We now re-run all of the computations using another basis: $\mathcal{B} = \{x_1, x_2, s_3\}$. We start by obtaining the final tableau:

x_1	x_2	s_1	s_2	s_3	rhs
0	0	3	$5/2$	0	42
1	0	1	0	0	4
0	1	0	$1/2$	0	6
0	0	-3	-1	1	-6

$$\mathcal{B} = \{x_1, x_2, s_3\} \quad \mathbf{c}_B^T = [3, 5, 0] \quad \mathbf{y}^T = \mathbf{c}_B^T B^{-1} = [3, 5/2, 0]^T$$

Now we have:

x_1	x_2	s_1	s_2	s_3
4	6	0	0	-6

e_1	e_2	y_1	y_2	y_3
0	0	3	$5/2$	0

This is dual feasible, but primal infeasible.

As the final computation, consider the basis $\{x_1, x_2, s_1\}$.

x_1	x_2	s_1	s_2	s_3	rhs
0	0	0	3/2	1	36
1	0	0	$-1/3$	1/3	2
0	1	0	1/2	0	6
0	0	1	1/3	$-1/3$	2

$$\mathcal{B} = \{x_1, x_2, s_1\}$$
$$\mathbf{c}_B^T = [3, 5, 0]$$
$$\mathbf{y}^T = \mathbf{c}_B^T B^{-1} = [0, 3/2, 1]^T$$

Now we have:

x_1	x_2	s_1	s_2	s_3
2	6	2	0	0

e_1	e_2	y_1	y_2	y_3
0	0	0	3/2	1

In this case, both the primal and dual are feasible, so this is the optimal solution for both.

Did you notice where the zeros appear? If you were to dot the solution to the primal with the solution to the dual, you **always get zero**. This is what is meant by "complementary slackness".