Goal Programming Example

Recall the football/soap opera problem.
We want to solve the linear program, given the following goals, and given the budget constraint:

- Goal 1: Get at least 40 HIM.
- Goal 2: Get at least 60 LIP.
- Goal 3: Get at least 35 HIW.

This translates to 3 “Row 0's”

$$\begin{aligned}
\min \quad & z = P_1 s_1 \\
\min \quad & z = \qquad\quad P_2 s_2 \\
\min \quad & z = \qquad\qquad\qquad P_3 s_3
\end{aligned}$$

with constraints:

$$\begin{array}{rll}
7x_1 + 3x_2 + s_1 - e_1 & = 40 & \text{HIM} \\
10x_1 + 5x_2 + s_2 - e_2 & = 60 & \text{LIP} \\
5x_1 + 4x_2 + s_3 - e_3 & = 35 & \text{HIW} \\
100x_1 + 60x_2 & \leq 600 & \text{Budget}
\end{array}$$

So that the (max) tableau:

x_1	x_2	e_1	e_2	e_3	s_1	s_2	s_3	s_4	rhs
0	0	0	0	0	P_1	0	0	0	0
0	0	0	0	0	0	P_2	0	0	0
0	0	0	0	0	0	0	P_3	0	0
7	3	-1	0	0	1	0	0	0	40
10	5	0	-1	0	0	1	0	0	60
5	4	0	0	-1	0	0	1	0	35
100	60	0	0	0	0	0	0	1	600

We need to "clean up" all the objective function rows so that we have columns of the identity.

x_1	x_2	e_1	e_2	e_3	s_1	s_2	s_3	s_4	rhs
$-7P_1$	$-3P_1$	P_1	0	0	0	0	0	0	$-40P_1$
$-10P_2$	$-5P_2$	0	P_2	0	0	0	0	0	$-60P_2$
$-5P_3$	$-4P_3$	0	0	P_3	0	0	0	0	$-35P_3$
7	3	-1	0	0	1	0	0	0	40
10	5	0	-1	0	0	1	0	0	60
5	4	0	0	-1	0	0	1	0	35
100	60	0	0	0	0	0	0	1	600

We focus on the Simplex Method using only the first “Row 0”. First, we pivot in Column 1 (aned first constraint row)

x_1	x_2	e_1	e_2	e_3	s_1	s_2	s_3	s_4	rhs
0	0	0	0	0	P_1	0	0	0	0
0	$-\frac{5}{7}P_2$	$-\frac{10}{7}P_2$	P_2	0	$\frac{10}{7}P_2$	0	0	0	$-\frac{20}{7}P_2$
0	$-\frac{13}{7}P_3$	$-\frac{5}{7}P_3$	0	P_3	$\frac{5}{7}P_3$	0	0	0	$-\frac{45}{7}P_3$
1	$3/7$	$-1/7$	0	0	$1/7$	0	0	0	$40/7$
0	$5/7$	$10/7$	-1	0	$-10/7$	1	0	0	$20/7$
0	$13/7$	$5/7$	0	-1	$-5/7$	0	1	0	$45/7$
0	$\frac{120}{7}$	$\frac{100}{7}$	0	0	$-\frac{100}{7}$	0	0	1	$\frac{200}{7}$

We focus on the Simplex Method using only the first “Row 0”. First, we pivot in Column 1 (and first constraint row).

x_1	x_2	e_1	e_2	e_3	s_1	s_2	s_3	s_4	rhs
0	0	0	0	0	P_1	0	0	0	0
0	$-\frac{5}{7}P_2$	$-\frac{10}{7}P_2$	P_2	0	$\frac{10}{7}P_2$	0	0	0	$-\frac{20}{7}P_2$
0	$-\frac{13}{7}P_3$	$-\frac{5}{7}P_3$	0	P_3	$\frac{5}{7}P_3$	0	0	0	$-\frac{45}{7}P_3$
1	$3/7$	$-1/7$	0	0	$1/7$	0	0	0	$40/7$
0	$5/7$	$10/7$	-1	0	$-10/7$	1	0	0	$20/7$
0	$13/7$	$5/7$	0	-1	$-5/7$	0	1	0	$45/7$
0	$\frac{120}{7}$	$\frac{100}{7}$	0	0	$-\frac{100}{7}$	0	0	1	$\frac{200}{7}$

We focus on the Simplex Method using only the first "Row 0". First, we pivot in Column 1 (and first constraint row). Pivot in Column 3, and after Ratio Test, use the second constraint (there was a tie between the second and last).

x_1	x_2	e_1	e_2	e_3	s_1	s_2	s_3	s_4	rhs
0	0	0	0	0	P_1	0	0	0	0
0	0	0	0	0	0	P_2	0	0	0
0	$-\frac{3}{2}P_3$	0	$-\frac{1}{2}P_3$	P_3	0	$\frac{1}{2}P_3$	0	0	$-5P_3$
1	$\frac{1}{2}$	0	$-\frac{1}{10}$	0	0	$\frac{1}{10}$	0	0	6
0	$\frac{1}{2}$	1	$-\frac{7}{10}$	0	-1	$\frac{7}{10}$	0	0	2
0	$\frac{3}{2}$	0	$\frac{1}{2}$	-1	0	$-\frac{1}{2}$	1	0	5
0	10	0	10	0	0	-10	0	1	0

We will now work with our third goal. Pivot in Column 2, last row.

$$
\begin{array}{c}
\begin{array}{cc|ccc|cccc|c}
x_1 & x_2 & e_1 & e_2 & e_3 & s_1 & s_2 & s_3 & s_4 & \text{rhs}
\end{array} \\
\left[\begin{array}{cc|ccc|cccc|c}
0 & 0 & 0 & 0 & 0 & P1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & P2 & 0 & 0 & 0 \\
0 & 0 & 0 & P3 & P3 & 0 & -P3 & 0 & \frac{3}{20}\,P3 & -5\,P3 \\
\hline
1 & 0 & 0 & -\frac{3}{5} & 0 & 0 & \frac{3}{5} & 0 & -\frac{1}{20} & 6 \\
0 & 0 & 1 & -\frac{6}{5} & 0 & -1 & \frac{6}{5} & 0 & -\frac{1}{20} & 2 \\
0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & -\frac{3}{20} & 5 \\
0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & \frac{1}{10} & 0
\end{array}\right]
\end{array}
$$

$$
\left[\begin{array}{cc|ccc|cccc|c}
x_1 & x_2 & e_1 & e_2 & e_3 & s_1 & s_2 & s_3 & s_4 & \text{rhs} \\
0 & 0 & 0 & 0 & 0 & P1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & P2 & 0 & 0 & 0 \\
0 & 0 & 0 & P3 & P3 & 0 & -P3 & 0 & \frac{3}{20}\,P3 & -5\,P3 \\
\hline
1 & 0 & 0 & -\frac{3}{5} & 0 & 0 & \frac{3}{5} & 0 & -\frac{1}{20} & 6 \\
0 & 0 & 1 & -\frac{6}{5} & 0 & -1 & \frac{6}{5} & 0 & -\frac{1}{20} & 2 \\
0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & -\frac{3}{20} & 5 \\
0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & \frac{1}{10} & 0
\end{array}\right]
$$

This is the final tableau. We were unable to meet Goal 3 ($S_3 = 5$), but we did meet goals 1 and 2: Use all ad time in football (6 units), and no time in soaps (0).

(Exercise 4, 4.16). We have two products, and we have 32 hours of labor available, a goal of 48 profit, and some demand.

$$\begin{array}{rrrrl} 4x_1 & +2x_2 & +s_1 & -e_1 & = 32 \text{ Labor} \\ x_1 & & +s_2 & -e_2 & = 7 \text{ Demand 1} \\ & x_2 & +s_3 & -e_3 & = 10 \text{ Demand 2} \\ 4x_1 & +2x_2 & +s_4 & -e_4 & = 48 \text{ Budget goal} \end{array}$$

- Goal 1: Avoid underutilization of labor.
- Goal 2: Meet demand for product 1.
- Goal 3: Meet demand for product 2.
- Goal 4: Do not use any overtime.

From our equations, we get the tableau (max):

$$\left[\begin{array}{cc|ccc|ccc|c}
x_1 & x_2 & s_1 & s_2 & s_3 & e_1 & e_2 & e_3 & \text{rhs} \\
\hline
0 & 0 & P_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & P_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & P_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & P_4 & 0 & 0 & 0 \\
\hline
4 & 2 & 1 & 0 & 0 & -1 & 0 & 0 & 32 \\
1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 7 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 10
\end{array}\right]$$

In this case, we'll solve in LINGO.

In LINGO, we'll minimize s_1 first:

```
min=s1;
4*x1+2*x2+s1-e1=32;
x1+s2-e2=7;
x2+s3-e3=10;
4*x1+2*x2+s4-e4=48;
```

In LINGO, we'll minimize s_1 first:

```
min=s1;
4*x1+2*x2+s1-e1=32;
x1+s2-e2=7;
x2+s3-e3=10;
4*x1+2*x2+s4-e4=48;
```

LINGO returns $x_1 = 0$ and $x_2 = 16$. Now we bring in the second constraint:

Set $s_1 = 0$ as a new constraint, and minimize s_2:

```
min=s2;
4*x1+2*x2+s1-e1=32;
x1+s2-e2=7;
x2+s3-e3=10;
4*x1+2*x2+s4-e4=48;
s1=0;
```

Set $s_1 = 0$ as a new constraint, and minimize s_2:

```
min=s2;
4*x1+2*x2+s1-e1=32;
x1+s2-e2=7;
x2+s3-e3=10;
4*x1+2*x2+s4-e4=48;
s1=0;
```

LINGO returns $x_1 = 7$ and $x_2 = 2$. Now bring in the third constraint:

Set $s_2 = 0$ as a new constraint, and minimize s_3:

```
min=s3;
4*x1+2*x2+s1-e1=32;
x1+s2-e2=7;
x2+s3-e3=10;
4*x1+2*x2+s4-e4=48;
s1=0;
s2=0;
```

Set $s_2 = 0$ as a new constraint, and minimize s_3:

```
min=s3;
4*x1+2*x2+s1-e1=32;
x1+s2-e2=7;
x2+s3-e3=10;
4*x1+2*x2+s4-e4=48;
s1=0;
s2=0;
```

LINGO returns $x_1 = 7$ and $x_2 = 10$. At this stage, we won't be able to drive e_1 to zero, and this is our optimal solution.